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FOUNDATIONS

★Ridder, J. **Sur quelques logiques multivalentes.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 728–730 (1949).

An abstract calculus of propositions is described which generalizes a calculus of S. A. Kiss. The system is shown to be consistent. It has an interpretation with $2^{\aleph_0} \times 2^{\aleph_0}$ elements in terms of a kind of product of two Boolean algebras. This leads the author to describe it as a many-valued logic. It is stated that one of the systems of C. I. Lewis and also the restricted functional calculus of Hilbert and Ackermann may be treated in the same way. The results reported on here have previously been published [Nederl. Akad. Wetensch., Proc. **51**, 670–680, 836–845 = Indagationes Math. **10**, 221–231, 264–273 (1948); these Rev. **10**, 230].

O. Frink (State College, Pa.).

Halldén, Sören. An analogy in modal logic to the Lesniewski-Mihailescu theorem. Norsk Mat. Tidsskr. **31**, 4–9 (1949).

A certain group of Polish and Romanian logicians have been interested in logical systems in which equivalence is taken as a primitive idea. The reason is that, from the more or less nominalistic standpoint of S. Leśniewski [Fund. Math. **14**, 1–81 (1929)], the equivalence connective is needed as a primitive in connection with definitions. Also it was discovered by Tarski [Fund. Math. **4**, 196–200 (1923)] that all the other propositional connectives could be defined in terms of equivalence provided quantifications were allowed over all binary truth functions. Accordingly Leśniewski [loc. cit.] and Mihailescu [Ann. Sci. Univ. Jassy. Partie I. **23**, 368–408 (1937), in particular, pp. 388–403] studied the set of formulas of ordinary propositional algebra which are expressible in terms of equivalence in the ordinary way. They showed that such a formula is a tautology if and only if every variable appears in it an even number of times and the number of negations is even. The present author solves the same problem for the Lewis algebra S5 of strict implication [see C. I. Lewis and C. H. Langford, Symbolic Logic, Century, New York, 1932, p. 501]. He shows that if P is constructed by the operations of strict and material equivalence, negation, and possibility from constituents of the form $\Diamond Q$, if each such constituent occurs an even number of times, and if the number of negations outside of the formulas Q is even, then P is a theorem of S5; the converse hold if each Q is a variable or the negation of a variable.

H. B. Curry (State College, Pa.).

Destouches-Février, Paulette. Le calcul des constructions. C. R. Acad. Sci. Paris **227**, 1192–1193 (1948).

Postulates and rules of procedure are given for an abstract calculus of constructions, analogous to a calculus of propositions. Letters represent constructions to be realized. A symbol of a unary operation, signifying that the construction which follows has been realized, corresponds to the assertion symbol in logic. There are also symbols for three binary operations, conjunction, disjunction, and an analogue of implication. No negation operation is present, however, since the negation of a construction is not a construction. The resulting system resembles somewhat the minimal calculus of Johansson. Applications to geometry and to the logical foundations of mathematics are suggested.

O. Frink (State College, Pa.).

Destouches-Février, Paulette. Connexions entre les calculs des constructions, des problèmes, des propositions. C. R. Acad. Sci. Paris **228**, 31–33 (1949).

Corresponding to the author's calculus of constructions [see the preceding review] there is a calculus of problems, and there is also a calculus of propositions of special type $p(a)$ or $p(x)$, propositions depending on a constant a or a variable x. A negation operation may be introduced into this calculus of propositions by adding a symbol standing for a contradiction.

O. Frink (State College, Pa.).

★Heredenduin, P. G. J. **The constructive method.** Library of the Tenth International Congress of Philosophy, Amsterdam, August 11–18, 1948, Vol. I, Proceedings of the Congress, pp. 749–751 (1949).

The author exhibits a formal deductive system called a constructive system, which is intended to formalize the informal but constructive reasoning used, for example, by Gentzen in proving the consistency of a portion of number theory, or in similar proofs of consistency of other formal axiomatic systems. The author's system has seven rules of truth and eight rules of derivation, stated explicitly. Instead of axioms it has rules of definition of three kinds, for classes, relations, and operations. As illustrations of what he means by rules of definition, the author states four such rules, one for a certain class, one for a certain operation, and two for relations (the identity and predecessor relations). These rules are in terms of two basic undefined operations, one unary and the other binary, on two basic elements. The resulting "constructive system," which is not developed in detail, would seem to be more elaborate than the formal axiomatic systems to which it is to be applied.

O. Frink (State College, Pa.).

Giorgi, Giovanni. A proposito di alcune discussioni recenti sui problemi della logica deduttiva. Boll. Un. Mat. Ital. (3) **3**, 256–259 (1948).

The author discusses critically recent notes of E. Carruccio concerning the consistency of logical systems, Gödel's theorem, and undecidable propositions. *O. Frink.*

Lafleur, Laurence J. Ambiguities in the Schröder-Bernstein theorem. Scripta Math. **13**, 169–175 (1947).

For a critical review cf. J. Symbolic Logic **13**, 158 (1948).

Riabouchinsky, Dimitri. Le point euclidien et le point dimensionnel. C. R. Acad. Sci. Paris **227**, 885–887 (1948).

The author uses his peculiar notion of the "origin" of a number, according to which numbers otherwise the same are different if they have different "origins," to explain away the paradox that an infinite set may apparently be in one-to-one correspondence with a proper subset of itself. For this purpose he introduces a distinction between "Euclidean" points, which are ordinary points, and "dimensional" points, which result from reducing to zero the diameter of a finite interval.

O. Frink (State College, Pa.).

Riabouchinsky, Dimitri. Sur les antinomies de la théorie des ensembles. C. R. Acad. Sci. Paris **227**, 1315–1317 (1948).

The author maintains that infinite sets are really not equivalent to proper subsets of themselves, and the whole is always actually greater than any of its proper parts. This results from an extension of his notion of the "origin" of a number to the number ∞. He also repeats a demonstration of a previous "theorem" of his which states that the set of all (unending) decimal fractions is denumerable.

O. Frink (State College, Pa.).

Morse, Marston. Equilibria in nature—stable and unstable. Proc. Amer. Philos. Soc. **93**, 222–225 (1949).

ALGEBRA

Wendelin, Hermann. Verallgemeinerung eines Interpolationsproblems. Arch. Math. **1**, 278–281 (1949).

In order that a polynomial $c_0x^n+c_1x^{n-1}+\cdots+c_n$ (in a field of characteristic 0) (1) assumes given values for n given values of x and (2) c_k is given for a given k it is necessary and sufficient that the kth elementary function of the given numbers should not vanish.

O. Todd-Taussky (Washington, D. C.).

Robinson, R. M. A note on linear equations. Amer. Math. Monthly **56**, 251 (1949).

***Arf, Cahit. Sur la définition des déterminants.** Université d'Istanbul. Faculté des Sciences. Recueil de mémoires commémorant la pose de la première pierre des Nouveaux Instituts de la Faculté des Sciences, pp. 9–20, Istanbul, 1948.

Let $U_n=(u_{rs})$ be a matrix whose elements are indeterminates and let U_h be the principal h-rowed minor matrix in the upper left position. The author defines a scalar function $F^h(U_h)$ which is shown to factor in the form

$$F^h(U_h)=D^{2h-2}(U_1)\cdot D^{2h-4}(U_2)\cdot\cdots\cdot D^2(U_{h-2})\cdot D(U_h).$$

The polynomial $D(U_n)$ thus recursively defined proves to be the determinant. *C. C. MacDuffee* (Madison, Wis.).

Banerjee, K. S. On the construction of Hadamard matrices. Science and Culture **14**, 434–435 (1949).

The author discusses the methods of construction of Hadamard matrices given by Plackett and Burman [Biometrika **33**, 305–325 (1946); these Rev. **8**, 44]. He shows that they are related to R. C. Bose's construction of symmetrical incomplete balanced block designs [Ann. Eugenics **9**, 353–399 (1939); these Rev. **1**, 199] with $4\lambda+3=p^n$ varieties, where p is a prime and $v=2^{n+1}-1$, $\lambda=2^{n-1}$.

H. B. Mann (Berkeley, Calif.).

Goheen, H. E. On a lemma of Stieltjes on matrices. Amer. Math. Monthly **56**, 328–329 (1949).

The author gives a simplified proof, by the method of complete induction, of the following theorem established by J. L. Mosak [General Equilibrium Theory in International Trade, Cowles Commission for Research in Economics, monograph no. 7, Principia Press, Bloomington, Ind., 1944, pp. 49–51; these Rev. **6**, 238]. If all the principal minors of a matrix are positive and all the elements off its main diagonal are negative, then all the elements of its inverse are positive. This theorem includes the following result of Stieltjes [see Acta Math. **9**, 385–400 (1887) =Oeuvres Complètes, v. 2, pp. 73–88]: If the elements off the principal diagonal of the symmetric matrix of a positive definite quadratic form are all negative, then all the elements of the inverse of that matrix are positive.

G. B. Price (Lawrence, Kan.).

Weizel, W. Hauptachsentransformation von Vierpolmatrizen und ihre Anwendung. Arch. Elektrotechnik **33**, 196–201 (1939).

Wegner, Udo. Remarque sur les valeurs propres des matrices. C. R. Acad. Sci. Paris **228**, 1200 (1949).

Corresponding to a characteristic root of the matrix A whose real part is negative, there is a characteristic root of $B=[A-I]^{-1}[A+I]$ within the unit circle, and vice versa. It is easy to determine if the absolute value of the characteristic root of B of maximal absolute value is less than 1, and thus to determine the stability of a linear dynamical system of matrix A. *C. C. MacDuffee* (Madison, Wis.).

Durand, Émile. Propriétés et applications de 4 matrices nouvelles reliées aux matrices de Dirac. J. Math. Pures Appl. (9) **28**, 1–33 (1949).

This paper has been essentially covered by three preliminary notes [C. R. Acad. Sci. Paris **220**, 517–520 (1945); **225**, 280–282, 375–377 (1947); these Rev. **7**, 181; **9**, 75, 76].

C. C. MacDuffee (Madison, Wis.).

Dieudonné, Jean. Sur une généralisation du groupe orthogonal à quatre variables. Arch. Math. **1**, 282–287 (1949).

Let K be a field and let K^{n^2} be the set of all matrices over K. Consider K^{n^2} as a vector space of dimension n^2. It is proved in the paper that a semi-linear transformation of K^{n^2} onto itself leaving $\det X=0$ invariant is of the form $X=PX^\sigma Q$ or $X=PX'^\sigma Q$, where X' is the transposed matrix of X and P and Q are nonsingular and σ is an automorphism of the field. *L. K. Hua* (Urbana, Ill.).

Brown, Ferdinand L. Remarks concerning tri-operational algebra. III. Rep. Math. Colloquium (2) **8**, 61–67 (1948).

[For parts I and II cf. same Rep. (2) **5–6**, 11–15 (1944); **7**, 61–64 (1946); these Rev. **6**, 143; **8**, 61.] The author seeks polynomials $f(x)$ with coefficients in $GF(p^n)$ which effect a permutation on the elements of $R=GF[p,x]/\phi(x)$. Let $f_r(x)$ denote a polynomial of degree not exceeding p^n-1 which

effects a permutation of $GF(p^n)$. Then for $\phi(x)=x^{p^n}-x$, $f(x)=f_r(x)$. For $\phi(x)=(x^{p^n}-x)^m$, $m>1$, $f(x)=f_r(x)$ if and only if $f_r'(a)\neq 0$ for any $a\varepsilon GF(p^n)$. A more general $\phi(x)$ is also treated. *L. Carlitz* (Durham, N. C.).

Littlewood, D. E. Invariant theory under orthogonal groups. Proc. London Math. Soc. (2) **50**, 349–379 (1948).

This paper continues a programme of applying the method of generating functions, based upon the theory of Schur's characteristics, to elaborate the systems of concomitants belonging to specified ground forms under linear transformations comprised in the orthogonal and rotational groups [cf. Philos. Trans. Roy. Soc. London. Ser. A. **239**, 387–417 (1944); these Rev. **7**, 6]. An essential simplification of earlier procedure has been secured by adopting for algebra the classical methods of Sylvester and of Klein, whereby an accurate correspondence between an orthogonal space and an auxiliary space is established. The author establishes a close connexion between the generating function for concomitants in the ternary orthogonal group and that for binary concomitants. The invariant theory of binary forms of even order, as developed classically, for projective transformations, has its precise counterpart in that for ternary forms of half this order in the orthogonal group. By utilizing ternary spinors binary forms of both odd and even order come into play. An appeal to the complete system of the binary sextic shows the isomorphism between this and the system for a ternary orthogonal cubic.

The quaternary orthogonal case is considered: (1) the proper or rotation group, with positive determinant, and (2) the full group, comprising also a negative determinant. The quaternary orthogonal quadratic is shewn to march in step with the double binary form of order (2, 2). Details of 14 concomitants up to degree four agree with the system first found by Peano (the remaining 4 forms in the original system being of degree 5 or 6). Several other systems are calculated, completely reduced up to the degree attained, in some cases degree 7, for both types of orthogonal group. Quaternary spinors, chirality, the Riemann-Christoffel tensor, the quadratic line complex enter and illustrate the discussion.

For the quinary case the orthogonal group is isomorphic with the quaternary symplectic group: for example, the quinary linear complex has the quaternary symplectic quadratic for counterpart. Various systems up to degree four are calculated. For the senary case a 2:1 isomorphism is established between the 6-dimensional rotation group and the 4-dimensional unitary group, by means of the correspondence between their Schur characters, including spin characters. This throws light upon, and improves, earlier discussions by the reviewer based upon the transformation, given by Klein, of lines in three-fold space to points of a quadric primal in five-fold space. Further applications with both seven and eight variables are made.

H. W. Turnbull (St. Andrews).

Abstract Algebra

***van der Waerden, B. L. Modern Algebra. Vol. I. Translated from the second revised German edition by Fred Blum. With revisions and additions by the author.** Frederick Ungar Publishing Co., New York, N. Y., 1949. xii+264 pp. $5.50.

The German edition was published by Springer, Berlin, 1937. The present edition has "restored the original way of introducing polynomials independent of hypercomplex number systems" and "treated in greater detail the valuations of algebraic number fields."

Ballieu, Robert. Géométrie et algèbre. Rev. Questions Sci. (5) **10**, 209–222 (1949).

A first introduction to the algebraic structures (fields and lattices) associated with abstract projective geometries.

D. B. Scott (London).

Krishnan, V. S. Extensions of partially ordered sets. II. Constructions. J. Indian Math. Soc. (N.S.) **12**, 89–106 (1948).

[For part I, see the same J. (N.S.) **11**, 49–68 (1947); these Rev. **10**, 95.] If K is a partially ordered set with operations θ_i, then L is a superset to $\phi=\{\theta_i\}$ of K if L is closed under ϕ and has a subset K' isomorphic to K. Call L a supersystem for ϕ of a subset $K'\subset L$ if whenever $A\subset K'$ and $\theta_i(A)$ exists in K' then $\theta_i(A)$ is preserved in L. Call L an extension to ϕ of K if L is a superset to ϕ of K and a supersystem for ϕ of K'. An extension L to ϕ of K in which (1) every element of L is "required" by ϕ and K, (2) for every two distinct elements of L there is an element of K contained in one but not both, or dually, and (3) the elements of L can be assigned representations such that the relation between every pair of them is "required" in all (a) extensions, (b) supersets to ϕ of K, is called a (a) canonical, (b) strong canonical extension to ϕ of K. Let $\prod, \sum$ denote unrestricted meet and join; P, S meet and join complementation.

Theorem: for a partially ordered set with units, canonical extensions (sometimes strong) exist to (0); to $(0, I)$; to all subsets of $(0, I, \cup, \sum, \cap, \prod)$ containing $\sum$ or $\prod$; to subsets of $(0, I, \cap, \prod, P)$ containing $(0, \prod, P)$; and dually. For a partially ordered set (units unnecessary), canonical extensions exist to (0); to $(0, I)$; to subsets of $(0, I, \cup, \sum, \cap, \prod)$ containing $(0, \sum)$ or $(I, \prod)$ or $(\sum, \prod)$; to subsets of $(\cup, \sum, I)$ containing $\sum$; and dually. The author asserts that for all other subsets of the operations mentioned plus distributive meets and joins (restricted or not), canonical extensions need not exist. However, example 5.51 appears invalid: K' is not isomorphic to K. *P. M. Whitman.*

Birkhoff, Garrett, and Whitman, Philip M. Representation of Jordan and Lie algebras. Trans. Amer. Math. Soc. **65**, 116–136 (1949).

Let X be either a Lie algebra or a Jordan algebra. To imbed X into an associative algebra A is to find a linear mapping f of X into A such that, if we denote by $*$ the multiplicative law of composition in X, then $f(x*y)$ is $f(x)f(y)-f(y)f(x)$ if X is a Lie algebra and $\frac{1}{2}(f(x)f(y)+f(y)f(x))$ if X is a Jordan algebra. There always exists a universal imbedding of X, i.e., an imbedding such that every other one may be obtained from it by applying a suitable homomorphism. If f is a universal imbedding, then the associative algebra generated by $f(X)$ is called the universal algebra of X.

In the case of the Lie algebra of a Lie group, it is shown that the infinitesimal transformations of the group yield the universal imbedding. It is also proved that every left ideal in the universal algebra A has a finite set of generators [the proof which is given of this fact is, however, difficult to understand, because it is entirely based on an undefined notion of homogeneity in the algebra A; the reviewer assumes that it is meant that the elements of A are to be represented as linear combinations of monomials of the type

described in theorem 2 and that the degrees have to be estimated in this representation; however, it is not proved that the representation is unique].

The authors show that, if X is a finite dimensional Jordan algebra, then the universal algebra of X is itself of finite dimension (a major difference from the case of Lie algebras). The universal imbeddings of the following Jordan algebras are explicitly determined: the Jordan algebra of all matrices of degree n; the algebra of symmetric matrices of degree n; the algebra of Hermitian matrices of degree n over the field of complex numbers; the algebra of Hermitian matrices over the quaternions. These results are contained as special cases in the results of F. D. Jacobson and N. Jacobson [cf. the paper reviewed below], but are anterior to them.

The relationship which exists between a Lie algebra of matrices and the Lie group of which it is the Lie algebra is extended to Jordan algebras. Let U be a neighborhood of the unit matrix which is the topological image under the exponential mapping of some compact neighborhood of the zero matrix. The authors call "local power family" a locally compact set S of matrices such that, for a suitable choice of U, $S \cap U$ is the image of a subset of V which is the intersection of V with a set of rays issued from the origin. The matrices Y on these rays are then called the infinitesimal matrices of S. It is proved that a necessary and sufficient condition for the infinitesimal matrices of S to form a Jordan algebra is for S to be locally convex; S then coincides on a neighborhood of the unit matrix I with the set of matrices $I+Y$, where Y is in the Jordan algebra. *C. Chevalley.*

Jacobson, F. D., and Jacobson, N. Classification and representation of semi-simple Jordan algebras. Trans. Amer. Math. Soc. 65, 141–169 (1949).

G. Birkhoff and P. M. Whitman have proved that any (finite dimensional) Jordan algebra X admits a universal imbedding in an associative algebra U which is of finite dimension [cf. the preceding review]. The problem of finding all imbeddings of X (and in particular the representations of X, which are the imbeddings in full matrix algebras) is therefore reduced to the determination of the universal algebra U and of its ideals. The Jordan algebra X is called special if its universal imbedding is an isomorphism. The present paper is mainly concerned with the determination of the universal imbeddings of the special simple Jordan algebras.

A. Albert has defined four classes of simple Jordan algebras over a field F, which are denoted by the letters A, B, C and D [Ann. of Math. (2) 48, 546–567 (1947); these Rev. 9, 77]. Moreover, he has proved that, if X is any central simple special Jordan algebra over a field F of characteristic 0, then there exists an overfield Z of F such that the algebra X_Z deduced from X by extending the basic field to Z is in one of the classes A, B, C or D. Whether F is of characteristic 0 or not, the authors of the present paper say that a Jordan algebra X is of type A, B, C or D if there exists an overfield Z of F such that X_Z is in the class A, B, C or D of Albert. For basic fields of characteristic 0, Albert has determined the structure of the algebras of type D [cf. loc. cit. above] and Kalisch the structure of the algebras of types B and C [Trans. Amer. Math. Soc. 61, 482–494 (1947); these Rev. 8, 561]. On the other hand, G. Birkhoff and P. M. Whitman have determined the universal algebras of certain classes of Jordan algebras [cf. the review above]. The authors determine completely all the Jordan algebras of type A, B, C or D over any field of characteristic not 2 and their universal algebras. It follows from the result of Albert quoted above and from this determination of the algebras of types A, B, C or D that a simple special Jordan algebra over a field F of characteristic 0 is isomorphic to one of the following algebras: (a) the algebra A_f of all elements of an associative simple algebra, in which we take $(a, b) \to \frac{1}{2}(ab+ba)$ for law of composition; (b) the algebra of symmetric elements of an associative algebra with respect to an involution of this algebra; (c) an algebra with a base $\{s_0, s_1, \cdots, s_n\}$ such that $s_0 s_i = s_i$, $s_i^2 = a_i s_0$ (with a_i in the basic field), $s_i s_j = 0$ $(1 \leqq i < j \leqq n)$. On the other hand, it turns out that the universal algebras of semi-simple Jordan algebras over fields of characteristic 0 are semisimple, which proves that the representations of these algebras are completely reducible. The authors prove that special simple Jordan algebras of different types cannot be isomorphic (apart from some restrictions on the orders), and they determine the derivations of the algebras of types A, B, C and D. *C. Chevalley* (New York, N. Y.).

Vranceanu, Georges. Sur la représentation linéaire des groupes de Lie intégrables. C. R. Acad. Sci. Paris 228, 885–886 (1949).

It is known that every Lie algebra has a faithful linear representation. The author studies the degree of the representation whose existence follows from the existence proof given by Cartan. He considers solvable Lie algebras of dimension r and he assumes that either (a) there are $r-1$ roots $\lambda_2, \cdots, \lambda_r$ different from 0 such that $\lambda_h + \lambda_k \neq \lambda_l$ $(2 \leqq h, k, l \leqq r)$, or (b) there exists a root $\lambda \neq 0$ of multiplicity $r-2$ and that 2λ is not a root. The degree is $r+1$ in case (a), and, "in general," $\frac{1}{2}(r-2)(r-3)$ in case (b). Canonical forms of structures of nilpotent algebras are also given in certain special cases. *C. Chevalley.*

THEORY OF GROUPS

***Scorza, Gaetano. Gruppi Astratti.** Edizioni Cremonese, Perella, Rome, 1942. vii+242 pp.

This book is concerned with abstract groups, finite or infinite; some results are stated for denumerable groups with the remark that the proof holds in general if the selection axiom is used. The book does not cover groups-with-operators nor group-characters and no attempt is made to connect with the theory of abstract algebras: there is, for example, no mention of congruences when dealing with normal subgroups. The multiplicative notation is used throughout. The notation distinguishes between a group and the set of all its elements, and the signs $+$, $\cap$ and $\geqq$ are used for set-theoretical union, intersection and inclusion. The definition of a group is by the associative law and the existence of a right-identity and right-inverses; no alternative forms are considered.

The elementary properties of orders and powers of elements, cosets and double cosets, isomorphisms and homomorphisms, holomorphs, normalizers and centralizers are dealt with in the usual way, and normal series are treated by the method of Zassenhaus [Lehrbuch der Gruppentheorie, v. 1, Teubner, Leipzig-Berlin, 1937]. The author then turns to the properties of successive adjoint groups, successive derived groups, the derived series, the upper

central series, the nucleus [German: Kern; R. Baer, Compositio Math. 1, 254–283 (1934)] and an analysis of Hamiltonian groups, which are shown to be of the form QA, where Q is the quaternion group and A is a special sort of commutative group.

One chapter is devoted to fundamental subgroups (i.e., normalizers of elements) and normocentres (i.e., centres of fundamental subgroups). Distinct fundamental subgroups are shown to have distinct centres, giving a one-one-one correspondence between fundamental subgroups, normocentres and sets of elements with the same normalizer. The number of fundamental subgroups is the scale [Italian: ceto] of the group. A fundamental subgroup is defied to be of genus 1 if it is minimal and of genus r if every fundamental subgroup contained in it is of genus less than r and if it has at least one fundamental subgroup of genus $r-1$ as a proper subgroup. The rank of a group is the greatest genus of any fundamental subgroup. A number of inequalities for the rank and the scale are found.

The fundamental structure theorem for commutative groups is proved using modules of n-sets of integers; and group-extensions are treated as in Zassenhaus [loc. cit.]. The definition of Sylow subgroup is generalized to apply to infinite groups: a Sylow p-subgroup is one the orders of all of whose elements are powers of p and which is maximal under this condition. A group has a Sylow p-subgroup if and only if it has elements whose orders are multiples of p, the Sylow p-subgroups are a class of conjugate subgroups, and their number is, if finite, congruent to 1 modulo p.

A group is called a p-group if the orders of its elements are powers of p: a finite p-group is then a p-group in the usual sense [P. Hall, Proc. London Math. Soc. (2) 36, 29–95 (1933)] and properties of p-groups, some taken from this paper, are given. This chapter contains Hall's theorem on solvable groups: if G is solvable and is of order mn, where m is prime to n, then it has a subgroup of order m, all subgroups of this order are conjugate, and every subgroup of G whose order is a factor of m is a subgroup of one of them. The chapter ends with Frobenius's theorem that the number of elements of a group of order n whose mth powers are in a given conjugacy class of order h is a multiple of (mh, n).

The last chapter deals with the displacement [Italian: traslato; German: Verlagerung] of a group, leading to two theorems due to Grün [J. Reine Angew. Math. 174, 1–14 (1935)] and one of Burnside [Proc. London Math. Soc. (1) 33, 257–268 (1901)] about Sylow subgroups.

The book is very clearly written (which makes for easy reading) and is adequately indexed. *H. A. Thurston.*

Buerger, M. J. Crystallographic symmetry in reciprocal space. Proc. Nat. Acad. Sci. U. S. A. 35, 198–201 (1949).

Es wird untersucht, wie sich das Auftreten eines Symmetrieelementes im Ausgangsraum auf den reziproken Raum auswirkt, und zwar zunächst für Symmetrieelemente durch den Ursprung, sodann für die anderen: Stellung und Art bleiben erhalten, während die Phase durch die Translation bestimmt wird. *J. J. Burckhardt* (Zürich).

Higman, G. Note on a theorem of R. Baer. Proc. Cambridge Philos. Soc. 45, 321–327 (1949).

The author proves the existence of an infinite sequence of integers $A(0)=1, A(1)=3, A(2)=13, \cdots$ with the following properties. If p is a prime number, if i is a positive integer such that $(p-1)A(m)<p^i-1$, and if j is some positive integer, then there exists a finite group G such that $1=G^{p^i}$ and such that the mth derived subgroup of G has class exceeding j. This result generalizes one obtained by Baer [Bull. Amer. Math. Soc. 50, 143–160 (1944), in particular, theorem 2 on p. 154; these Rev. 5, 227]. The construction is effected by means of a discussion of homomorphisms of some special partially ordered sets. *R. Baer.*

Grün, Otto. Über die genaue Anzahl von Elementen gegebener Ordnung in einer endlichen Gruppe. Math. Nachr. 1, 342–344 (1948).

Let G be a group of finite order g and let n_d be the number of elements of G of exact order d. The author obtains the following congruence connecting n_d, g and an arbitrary positive integer $m \leqq g$:

$$\sum_d \binom{g/d}{m/d} n_d \equiv 0 \pmod{g},$$

where the summation is over all d dividing (g, m). By rewriting the above in the form

$$\sum_d \binom{g/d-1}{m/d-1} n_d \equiv 0 \pmod{m},$$

and letting $m=1, 2, \cdots, g$, a set of congruences for the numbers n_d is obtained. Examples are given to show that there exist solutions of these congruences which do not arise from any group. *S. A. Jennings* (Vancouver, B. C.).

Čunihin, S. A. On Sylow-regular groups. Doklady Akad. Nauk SSSR (N.S.) 60, 773–774 (1948). (Russian)

Let g be the order of a group G, and let π be any collection of prime numbers. The group G is called π-Sylow-regular if G has, for every number m, $(m, g/m)=1$, all the prime factors of which are in π, a soluble subgroup of order m, and if moreover every two such subgroups are conjugate in G. If the successive factor groups in a composition series of a group H are π-Sylow-regular, then H is π-Sylow-regular. Every π-separable group is π-Sylow-regular. [Cf. same Doklady (N.S.) 59, 443–445 (1948); Gol'berg, same Doklady 64, 615–618 (1949); these Rev. 9, 492; 10, 505.] *L. F. Walton* (Santa Barbara, Calif.).

Zappa, Guido. Sui sottogruppi finiti dei gruppi di Hirsch. Giorn. Mat. Battaglini (4) 2(78), 55–70 (1948).

This is a continuation of two papers reviewed earlier [Rend. Sem. Mat. Univ. Padova 12, 1–11, 62–80 (1941); these Rev. 8, 252]. Definitions, etc. may be found in these reviews. The author first generalizes to a slightly wider class of S-group two theorems obtained previously for supersoluble S-groups. An S group is said to be of "species k" if the maximum number of independent generators of any infinite factor of a chief series of G is k, while G is of "type k" if all the factors (finite and infinite) in a minimal chief series have at most k independent generators. (A supersoluble S-group is of type 1, and species 1 therefore.) The following theorems are obtained. If the orders of the finite factors of a minimal chief series of an S-group of species k are not divisible by any prime less than or equal to $k+1$, then G has a finite characteristic subgroup consisting of all the elements of finite order in G, and the order of this subgroup is equal to the product of the orders of the finite factors of a minimal chief series. If G is an S-group of type 2, the elements of G of finite orders prime to 2 and 3 form a characteristic subgroup whose order equals the product of the orders prime to 2 and 3 of the finite factors of a minimal chief series having the maximum number of independent

generators in each factor. Finally, Sylow 2-groups for a special type of supersoluble S-group are discussed, and conditions are given to ensure that all elements of G of order a power of 2 are contained in a 2-group whose order is equal to the product of the orders of the factors of order 2 in a minimal chief series. *S. A. Jennings* (Vancouver, B. C.).

Černikov, S. N. On the theory of special p-groups. Doklady Akad. Nauk SSSR (N.S.) **63**, 11–14 (1948). (Russian)

Let G be a p-group having no finitely generated subgroup of infinite order, but having an infinite descending chain of subgroups. Then G has an infinite descending chain of Abelian subgroups. The proof is based on a study of the upper central series of G. See papers by the same author [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 199–214 (1939); **7(49)**, 35–64, 539–548 (1940); **18(60)**, 397–422 (1946); these Rev. **1**, 162; **2**, 5, 126; **8**, 311]. *J. L. Brenner.*

Černikov, S. N. On the theory of locally soluble groups with the minimal condition for subgroups. Doklady Akad. Nauk SSSR (N.S.) **65**, 21–24 (1949). (Russian)

Let $\mathfrak{M}$ denote the class of groups possessing the minimal condition for Abelian subgroups. The author generalizes his earlier conclusion that every infinite locally finite p-group in $\mathfrak{M}$ is special [cf. the preceding review] to obtain the following principal result. Every infinite locally solvable group in $\mathfrak{M}$ is the extension of a direct product of finitely many quasi-cyclic groups by a finite solvable group. Other theorems include: every locally solvable group in $\mathfrak{M}$ not containing an infinite Sylow subgroup is finite; a complete locally solvable group in $\mathfrak{M}$ is Abelian; a locally solvable group possesses the minimal condition for subgroups if and only if it belongs to $\mathfrak{M}$. *R. A. Good* (College Park, Md.).

Muhammedžan, H. H. On the theory of infinite groups with an ascending central series. Doklady Akad. Nauk SSSR (N.S.) **65**, 269–272 (1949). (Russian)

A "special" group is one in which (i) the minimal condition for subgroups holds and (ii) every proper subgroup is different from its normaliser. S. Černikov [Rec. Math. [Mat. Sbornik] N.S. **17(59)**, 105–130 (1945); these Rev. **7**, 411] has given examples of infinite special p-groups whose upper central series has only finite factors. In the present note the author proves that nonspecial groups with this property do not exist. Theorem: if the p-group $\mathfrak{G}$ has an ascending central series, and if all factors of its upper central series are finite, then $\mathfrak{G}$ is a special group. The idea of the proof cannot be indicated briefly. [Misprint: p. 271, line 8 from top, instead of the second $\mathfrak{Z}_0'$ read $\mathfrak{Z}_1'$.]

K. A. Hirsch (Newcastle-upon-Tyne).

Nielsen, Jakob. The commutator group of the free product of cyclic groups. Mat. Tidsskr. B. **1948**, 49–56 (1948). (Danish)

By an argument involving covering spaces, non-Euclidean area and groups of motion in the hyperbolic plane the author proves that the commutator subgroup of the free product of r finite cyclic groups of orders $m_1, \cdots, m_r$ is a free group of rank

$$1+m_1m_2\cdots m_r\cdot\left\{-1+\sum_{\lambda=1}^{r}(1-1/m_\lambda)\right\}.$$

The fact that this can also be proved purely algebraically from the Reidemeister-Schreier theorem is duly noted.

R. H. Fox (Princeton, N. J.).

Haimo, Franklin. A class of inverse limit groups. Amer. J. Math. **71**, 171–177 (1949).

The author has shown recently [Duke Math. J. **15**, 347–356 (1948); these Rev. **10**, 10] that even strong direct sums of the additive group of integers can never admit a topology which makes them compact groups. In the present paper this result appears in a different light. Let G be an Abelian group and G' be its subgroup of completely divisible elements, i.e., $G'=\bigcap_{n=1}^{\infty}nG$. Let G_∞ be the inverse limit group [for definitions see Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942, pp. 31–32, 54–56; these Rev. **4**, 84] of the set of quotient groups G/nG, $n=1, 2, \cdots$. Then G_∞ has no completely divisible elements other than 0, and G/G' is isomorphic to a subgroup of G_∞. If G has at least one topology which makes it a compact topological group, then it is shown that G/G' is isomorphic to the whole of G_∞. The author proves further that for the group I of integers this is not the case and hence the above result follows.

In the paper the theory is developed in much more generality. Instead of the groups nG, $n=1, 2, \cdots$, any directed system $\mathfrak{H}$ of subgroups of G is considered, where $\mathfrak{H}$ is indexed by a directed set A with partial order $<$ such that for $\alpha, \beta\in A$, $\alpha<\beta$, the corresponding subgroups satisfy $H_\alpha\supset H_\beta$. The system of quotient groups $\{G/H_\alpha\}$ is made in a natural manner into an inverse limit group which is denoted by $\mathrm{invlim}(G; \mathfrak{H})$ and replaces the group G_∞ above; the role of G' is taken by the group $H=\bigcap_{\alpha\in A}H_\alpha$. Then G/H is isomorphic to a subgroup of $\mathrm{invlim}(G; \mathfrak{H})$, viz., that formed by the "canonical" elements of $\mathrm{invlim}(G; \mathfrak{H})$ (these canonical elements can be characterized independently by coset properties) and every element of the group $\mathrm{invlim}(G; \mathfrak{H})$ is canonical, if G has at least one topology which makes it a compact topological group (with a slight restriction on the set $\mathfrak{H}$).

K. A. Hirsch (Newcastle-upon-Tyne).

Kar, S. C. Zur Analyse der Lorentzgruppe in algebraischer Darstellung. Bull. Calcutta Math. Soc. **40**, 163–172 (1948).

Derivation of the known result that a Lorentz-transformation can be written as a product of two Lorentz-transformations which correspond to an infinitesimal Lorentz-transformation and its complex conjugate. There is also a paragraph on the relation with the so-called Pauli-matrices.

F. I. Mautner (Cambridge, Mass.).

Banach, S. Remarques sur les groupes et les corps métriques. Studia Math. **10**, 178–181 (1948).

Let G be a complete metric space and as well a group in which xy is continuous in x and such that the set of (x, x^{-1}) for all x is closed in G^2. It is proposed that the passage from x to x^{-1} is continuous on G. Although the argument expressly uses left-continuity of xy it also involves a lemma whose proof seems to require right-continuity. Thus in case G is separable, the argument can be completed yielding the proof and result of D. Montgomery [Bull. Amer. Math. Soc. **42**, 879–882 (1936)]. The extension to the general case is based on the supposition that the smallest closed subgroup in G containing a given countable subgroup is separable. The proof of this on the present hypotheses presents difficulties which are passed over in the presentation, which was prepared by S. Hartman.

R. Arens (Los Angeles, Calif.).

Montgomery, Deane. Subgroups of locally compact groups. Amer. J. Math. **70**, 327–332 (1948).

Soit G un groupe convexe et localement compact. Alors le composant de l'identité de chaque entourage de l'identité engendre le groupe total G. Chaque sous-groupe fermé de G a une dimension plus basse que celle de G (si celle-ci est finie). Ce dernier théorème se démontre avec les méthodes combinatoires, dont l'auteur a fait usage à plusieurs reprises.

H. Freudenthal (Utrecht).

Gel'fand, I. M., and Naĭmark, M. A. The trace in fundamental and supplementary series of representations of the complex unimodular group. Doklady Akad. Nauk SSSR (N.S.) **61**, 9–11 (1948). (Russian)

[Pour les notations voir Mat. Sbornik N.S. **21(63)**, 405–434 (1947); Doklady Akad. Nauk SSSR (N.S.) **58**, 1577–1580 (1947); ces Rev. **9**, 328, 329; remplacer sur p. 329, première colonne, ligne 16 d'en bas la lettre k par K.] Les auteurs s'occupent des traces des représentations unitaires du groupe complexe unimodulaire. Ces traces sont regardées comme des fonctionnelles définies dans l'anneau de groupe (c'est-à-dire dans l'anneau des fonctions sommables, où le produit est la convolution de deux fonctions). Si U_g est une représentation de la série non-dégénérée et $x(g)$ est un élément de l'anneaux de groupe, la transformation $U_x f(z) = \int x(g) U_g f(z) d\mu(g)$ possède un noyeau intégral dont la trace est

$$\int x(g) \frac{\sum \chi(k_\sigma)}{D(g)} d\mu(g)$$

où la somme s'étend [comme celle qui a été citée en ces Rev. **9**, 328, deuxième colonne, dernière formule] sur toutes les permutations des valeurs propres de g sur la diagonale principale de g, et où $D(g)$ est les discriminant de l'équation caractéristique de g. Pour les séries dégénérés la formule est analogue, mais plus compliquée. On donne des critères nécessaires et suffisants pour l'équivalence de deux représentations.

H. Freudenthal (Utrecht).

Doss, Raouf. Sur l'immersion d'un semi-groupe dans un groupe. Bull. Sci. Math. (2) **72**, 139–150 (1948).

Let S be a semi-group, i.e., an associative system with cancellation on both sides. An element a of S is called regular on the left if it has a common left multiple with every other element b of S: $ra = sb$. The semi-group S is called quasi-regular on the left if, whenever two elements a, b of S have a common left multiple, $ra = sb$, it is possible to find r', s' in S such that $r'a = s'b$ and such that either r' or s' is regular on the left. The author shows that if S is quasi-regular on the left then S can be embedded in a group. The method of proof is to show by induction that the Malcev compatibility conditions are satisfied [A. Malcev, Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 331–336 (1939); these Rev. **2**, 7]. [The author's clear statement of these conditions is very welcome.] This embedding theorem contains as special cases those of Ore [Ann. of Math. (2) **32**, 463–477 (1931)] and M.-L. Dubreil-Jacotin [C. R. Acad. Sci. Paris **225**, 787–788 (1947); these Rev. **9**, 174].

A. H. Clifford (Baltimore, Md.).

Choudhury, A. C. Quasi-groups and nonassociative systems. I. Bull. Calcutta Math. Soc. **40**, 183–194 (1948).

The first section of this paper contains a number of elementary results on quasi-groups, particularly concerned with relative left and right identities (i.e., solutions x, y of $xa = a$ and $ay = a$ for given a). The second section is chiefly concerned with known properties of isotopy and homotopy as applied to quasi-groups. The final section is concerned with a nontransitive "ordering" relation $a \geqq b$ in an Abelian quasi-group, defined by the statement: $a \geqq b$ means $ab = b$. Owing to inconsistent use of the terminology, this section is difficult to follow.

D. Rees (Manchester).

NUMBER THEORY

Jarden, Dov. On the factorization of Fibonacci numbers. Riveon Lematematika **2**, 61–62 (1948). (Hebrew)

The author gives a list of linear congruences modulo $20n$ satisfied by every prime factor of the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ···. These results are said to be more explicit and more useful in factoring this sequence than those given originally by Lucas in 1878.

D. H. Lehmer (Berkeley, Calif.).

Yakovkin, M. V. Once more on a criterion for the irreducibility of polynomials. Doklady Akad. Nauk SSSR (N.S.) **64**, 771–774 (1949). (Russian)

The author proves the following criterion for the irreducibility of a polynomial. Let $f(x)$ be a polynomial of degree n with integral coefficients, and suppose that the real part of no root of $f(x)$ exceeds the rational number p_1/q_1, and that p_2/q_2 is a positive rational number such that the integer $\{q_1q_2/(q_1, q_2)\}^n f(p_1/q_1 + p_2/q_2)$ cannot be split into two factors each not less than

$$\left| \frac{(p_2q_1)^{m+1} - q_2^{m+1}}{(p_2q_1 - q_2)(q_1, q_2)^m} \right|, \qquad 1 \leqq m \leqq \tfrac{1}{2}n;$$

then there does not exist a polynomial with integral coefficients and degree s ($m \leqq s \leqq \frac{1}{2}n$) which is a factor of $f(x)$. This result is of the same type as theorem 2 of an earlier paper by the author [same Doklady (N.S.) **58**, 1915–1918 (1947); these Rev. **9**, 331].

R. A. Rankin.

Bell, E. T. A type of inversion of certain series. Duke Math. J. **15**, 79–85 (1948).

The general problem of inversion of finite series is treated from a viewpoint suggested by the general inversion formula of H. F. Baker [Proc. London Math. Soc. (1) **21**, 30–32 (1891)]. The result obtained is contained in the following statement. If $F_r(X_r, X'_{n-r}) = P(T) f_r(X_r, X'_{n-r})$ $(r = 0, 1, \cdots, t)$ then $f_r(X_r, X'_{n-r}) = P^{-1}(T) F_r(X_r, X'_{n-r})$ $(r = 0, 1, \cdots, t)$, where f_r are single valued functions whose values belong to a module; X_r is a subset of precisely r elements taken from an abstract set $x_1, x_2, \cdots, x_n$ and X'_{n-r} is the complementary subset; $P(U)$ is an arbitrary polynomial for which $P(0) = 1$ and $T^s = T_s$ is an operator defined by

$$T_s f_r(X_r, X_{n-r}) = f_{r-1}(X_{r-1}, X_{n-r+1})$$

if $x_s \subset X_r$ and 0 otherwise, where X_{r-1} is the result of deleting x_s from X_r; $P^{-1}(T)$ is the unique operator inverse to $P(T)$. Special cases include the principle of cross classification, Dedekind's inversion formula, L.C.M. inversion and their known generalisations.

D. H. Lehmer (Berkeley, Calif.).

Bell, E. T. Derived Diophantine systems. Revista Ci., Lima **50**, 69–84 (1948).

Let $P(X)$, $X=(X_1, \cdots, X_m)$ denote a system (one or more) of Diophantine equations whose complete solution is known explicitly, say $X=X'$. Let $D(x)$, $x=(x_1, \cdots, x_n)$ be a second Diophantine system such that when $x=x'$ ranges over the complete solution of $D(x)$, $X_i=f_i(x)$ ranges over the complete solution of $P(X)$. Then the complete solution of $D(x)$ is the solution of $f_i(x)=X_i'$; if $P(X)$ has no solutions, neither has $D(x)$. Several examples are given showing how to derive $D(x)$ from a system $P(X)$ which is separable in a sense defined in a previous paper [Trans. Amer. Math. Soc. **57**, 86–101 (1945); these Rev. **6**, 256]. For example, if $P(X)$ is $X^3+Y^3=AZ^3$, then $D(x)$ can be taken as $yz^2(4bw^3-a^2x^4y^3)=3u^3$, $ab=A$, where the $X_i=f_i(x)$ are $X=\frac{1}{2}(ax^2y^2z+u)$, $Y=\frac{1}{2}(ax^2y^2z-u)$, $Z=xyzw$. In this case $D(x)$ was obtained from $P(X)$ by treating the latter as a multiplicative equation $RS=AZ^3$, whose complete solution is $R=ax^2y^2z$, $S=byz^2w^3$, $A=ab$, $Z=xyzw$. The rest follows by identifying R and $X+Y$, and writing u for $X-Y$. By thus taking known equations for $P(X)$ the author solves the derived equations $D(x)$. [Reviewer's remark: (6.1) and (6.2) are incorrect, because their solutions corresponding to certain trivial solutions of their primitive were omitted; e.g., the equation in (6.2) has the solution $c=3$, all other symbols equal to one.] *I. Niven* (Eugene, Ore.).

Moessner, A. Alcune ricerche di teoria dei numeri e problemi diofantei. Boll. Mat. (5) **2**, 36–39 (1948).

Ten miscellaneous results on Diophantine equations are given. Parametric solutions in rational numbers are given of the equations

$$x^n+y^n=(x-y)^{n+1},$$
$$x+y=x^n+y^n,$$
$$x-y=x^3-y^3,$$
$$x^3+y^3=(x+y)^2,$$
$$x^2+y^2+z^2=(x+y-z)^2=w^4.$$

All integers X_i $(i=1, 2, \cdots, 6)$ less than 1000 in absolute value whose sum and sum of 5th powers both vanish are given. Other results involve the Tarry-Escott problem in special cases. A parametric ideal solution is given of the 7th degree problem. The reader should be on the alert for misprints. *D. H. Lehmer* (Berkeley, Calif.).

Gupta, Hansraj. On $N_s(r)$ in the Tarry-Escott problem. Proc. Nat. Inst. Sci. India **14**, 335–336 (1948).

The function $N_s(r)$ is the least value of s for which there exist q sets of s integers each, no set a mere permutation of another, which have the same sums of kth powers for $k=0, 1, \cdots, r$. The author proves that $N_s(r)\leqq(r^2+r+2)/2$, a result obtained also by E. M. Wright [Bull. Amer. Math. Soc. **54**, 755–757 (1948); these Rev. **10**, 101] by a similar argument. *D. H. Lehmer* (Berkeley, Calif.).

Hlaváček, Miloslav. Again on the solution of the equation $x^4-y^4=z^4-u^4$ in integers. Časopis Pěst. Mat. Fys. **73**, D43–D46 (1949). (Czech)

Denote by $\xi(v)$, $\eta(v)$, $\zeta(v)$ the quotients $\vartheta_i(v, \tau)/\vartheta_0(v, \tau)$ $(i=1, 2, 3)$ of the Jacobi theta-functions. If $\xi(v)$, $\eta(v)$, $\zeta(v)$ are rational, then $\xi((2k+1)v)=a_k/d_k$, $\eta((2k+1)v)=b_k/d_k$, $\zeta((2k+1)v)=c_k/d_k$ $(k=1, 2, 3, \cdots)$ are also rational, and the integers a_k, b_k, c_k, d_k therefore give a series of solutions of the equation of the title (since $\xi^4-\eta^4=1-\zeta^4$), for which $(a_k^2+d_k^2)/(c_k^2-b_k^2)$ is independent of k. See also Hlaváček [same Časopis **67**, D1–D4 (1938)], where an analogous result is obtained with the aid of the $\wp$-function. *V. Jarník* (Prague).

Weil, André. Numbers of solutions of equations in finite fields. Bull. Amer. Math. Soc. **55**, 497–508 (1949).

According to the author the purpose of this paper is to give an exposition of known results concerning the equation

$$a_0x_0^{m_0}+\cdots+a_rx_r^{m_r}=b, \quad a_i, x_i, b\in GF(p^n).$$

In addition to his own work he refers to recent papers by Davenport and Hasse and Hua and Vandiver as well as Gauss, Jacobi, Lebesgue and Stickelberger. The paper closes with some conjectures "concerning the number of solutions of equations over finite fields and their relation to the topological properties of the varieties defined by the corresponding equation over the field of complex numbers." *L. Carlitz* (Durham, N. C.).

Chowla, S. The last entry in Gauss's diary. Proc. Nat. Acad. Sci. U. S. A. **35**, 244–246 (1949).

The author proves the following two theorems by elementary methods: (I) $\sum_{x=1}^{p-1}((x^4-1)/p)=2\theta-2$, where p denotes a prime of the form $4n+1$, (m/p) is the Legendre symbol, θ is defined by $p=\theta^2+\phi^2$, and $\theta\equiv 1$ or 3 (mod 4), according as $p\equiv 5$ or 1 (mod 8); (II) $\sum_{x=1}^{p-1}((x^6-1)/p)=-4\theta-2$, where p denotes a prime of the form $12n+1$, θ is defined by $p=\theta^2+3\phi^2$, and $\theta\equiv 1$ (mod 3). Theorem I implies a theorem stated as a conjecture by Gauss in the last entry of his Tagebuch [Werke, vol. X_1, p. 571]. [Note by the reviewer. A. Weil has pointed out, in the paper reviewed above, that the first proof of Gauss's conjecture is due to Gauss himself, Werke, vol. II, p. 89.] *A. L. Whiteman* (Los Angeles, Calif.).

Sudan, Gabriel. Remarques sur les corps quadratiques de nombres. Acad. Roum. Bull. Sect. Sci. **28**, 593–603 (1946).

The author gives simple proofs for special cases of theorems of Dickson [Algebren und ihre Zahlentheorie, Füssli, Zurich, 1927, pp. 150–151], E. Berg [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **5**, 53–58 (1935)], and J. Fox Keston [Thesis, Yale University, 1935; Bull. Amer. Math. Soc. **41**, 186 (1935)] that no Euclidean algorithm exists in certain quadratic fields. Moreover, the author completes Euler's solution of the Diophantine equation $y^3=x^2+2$. However, his method is well known. [See, for instance, Pepin, J. Math. Pures Appl. (3) **1**, 317–372 (1875); Ivanoff, Intermédiaire Math. **6**, 47–48 (1899); Landau, ibid. **8**, 145–147 (1901).] The author's proof that $x+\sqrt{2}$ and $x-\sqrt{2}$ are relatively prime for odd x can be simplified since a common divisor must divide $2x$ and $2\sqrt{2}$, hence 2. *A. Brauer.*

Schneider, Theodor. Über eine Dysonsche Verschärfung des Siegel-Thueschen Satzes. Arch. Math. **1**, 288–295 (1949).

A simplified proof is given of a theorem due to the reviewer [Acta Math. **79**, 225–240 (1947); these Rev. **9**, 412], that for any algebraic number θ of degree n the inequality $|\theta-p/q|<q^{-\mu}$ has at most a finite number of solutions for every $\mu>(2n)^{\frac{1}{2}}$. The proof follows the methods used in an earlier paper of the author [J. Reine Angew. Math. **175**, 182–192 (1936)]. The complicated lemma on which the reviewer's proof was based turns out to be entirely unnecessary. Since the constant $(2n)^{\frac{1}{2}}$ is presumably far

from the best possible, it is of interest to note that each of the two proofs makes use of some information which the other ignores. The question arises, whether a better result might be obtained by combining the two methods.

F. J. Dyson (Birmingham).

Perron, Oskar. Diophantische Ungleichungen in imaginären quadratischen Körpern. Mat. Tidsskr. B. **1949**, 1–17 (1949).

The paper extends known results on the product of two homogeneous or nonhomogeneous linear forms to the fields of the title. Theorem 1. If $\alpha, \beta, \gamma, \delta$ are complex numbers with $|\alpha\delta-\beta\gamma|=1$, there are infinitely many pairs of integers $x, y \neq k(i\sqrt{D})$ for which $|(\alpha x+\beta y)(\gamma x+\delta y)| \leqq K$, where $K=(6D)^{\frac{1}{2}}/\pi$ for $D \not\equiv 3 \pmod 4$, $K=(6D)^{\frac{1}{2}}/(2\pi)$ for $D \equiv 3 \pmod 4$. Theorem 2. If $\alpha, \beta, \gamma, \delta, \rho, \sigma$ are complex numbers with $|\alpha\delta-\beta\gamma|=1$, there are integers $x, y \neq k(i\sqrt{D})$ for which $|(\alpha x+\beta y-\rho)(\gamma x+\delta y-\sigma)| \leqq L$, where

$$L=(\tfrac{1}{4}(1+D))^{\frac{1}{2}h}D^{\frac{1}{2}-\frac{1}{2}h}(\Delta+(6D)^{\frac{1}{2}}/h\pi).$$

Here $h=1$ or 2 according as $D \not\equiv 3$ or $D \equiv 3 \pmod 4$, and Δ is the smallest number such that each class of ideals in $k(i\sqrt{D})$ contains an ideal with norm not exceeding Δ. These theorems are not best possible. In the case of theorem 1 the author has shown that K can be replaced by $1/\sqrt{3}$ for $D=1$ and $1/13^{\frac{1}{4}}$ for $D=3$ [Math. Z. **35**, 563–578 (1932)] and by $1/\sqrt{2}$ for $D=2$ [Math. Z. **37**, 749–767 (1933)], but by no smaller constants. Concerning theorem 2, the author had proved earlier [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 159–165 (1947); these Rev. **9**, 569] that the best possible constant for $D=1$ is $\frac{1}{2}$; he proves in the present paper that the L above can be replaced by $\frac{3}{4}$ and $\frac{1}{4}$ in the cases $D=2$, $D=3$, but by no smaller numbers. The proofs given depend on Minkowski's theorem on lattice points in convex regions and the existence of integers of the field in question inside certain Cassini ovals in the complex plane.

W. J. LeVeque (Ann Arbor, Mich.).

Davenport, H. On indefinite ternary quadratic forms. Proc. London Math. Soc. (2) **51**, 145–160 (1949).

If $Q(x, y, z)$ denotes an indefinite ternary quadratic form with real coefficients and determinant $D \neq 0$, there exist integers x, y, z not all zero such that $|Q| \leqq (\frac{2}{7}|D|)^{\frac{1}{2}}$. Analogously to this result of Markov, the author proves that, if $D>0$, there exist integers x, y, z for which $0<Q \leqq (\frac{4}{9}D)^{\frac{1}{2}}$; if $D<0$ there exist integers x, y, z for which $0<Q \leqq (4|D|)^{\frac{1}{2}}$. Both theorems are best possible as is shown by the forms $x^2-\frac{1}{2}y^2-\frac{1}{2}z^2-\frac{3}{2}yz-\frac{3}{2}zx+\frac{3}{2}xy$, x^2+yz, respectively.

J. F. Koksma (Amsterdam).

Mullender, P. Lattice points in non-convex regions. II. Nederl. Akad. Wetensch., Proc. **51**, 1251–1261 = Indagationes Math. **10**, 395–405 (1948).

[For part I cf. same vol., 874–884 = Indagationes Math. **10**, 302–312 (1948); these Rev. **10**, 285.] The author considers the following problem. Given a closed region R in n-dimensional Euclidean space E, construct a region $K \subset E$, with the (largest possible) volume V_K, such that all points P_1-P_2, with P_1 and P_2 in K, are points of R. The author calls such a region K suitable. By a theorem of H. F. Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)] any lattice Λ, with $0<|\Delta| \leqq V_K$ (Δ denotes the determinant of Λ), then has a point other than the origin contained in R. In part A of this paper $R \subset E$ is the set of the points $(x_1, \cdots, x_n)$, with $|x_1 \cdots x_n| \leqq 1$. Using results of J. F. Koksma and B. Meulenbeld [same Proc. **45**, 256–262, 354–359, 471–478, 578–584 (1942); these Rev. **5**, 256] and L. J. Mordell [same Proc. **49**, 773–781, 782–792 = Indagationes Math. **8**, 476–484, 485–495 (1946); these Rev. **8**, 369] the author constructs a suitable region K (which is larger than an almost trivial one defined by $|x_1|+\cdots+|x_n| \leqq n/2$) and gives its volume. In part B the author constructs a suitable region K for the regions R defined as follows: let $F(x) \equiv F(x_1, \cdots, x_n)$ be a single-valued, twice differentiable function of $x_1, \cdots, x_n$ defined for all nonnegative values of the variables; let $F(x)>0$, $\partial F(x)/\partial x_j>0$ $(j=1, \cdots, n)$, $F(x)+F(y) \leqq F(x+y)$, $F(tx)=tF(x)$ for $x_1, \cdots, x_n>0$; $y_1, \cdots, y_n>0$; $t \geqq 0$; then R is defined as the set of points (x) satisfying the inequality $F(|x_1|, \cdots, |x_n|) \leqq F(1, \cdots, 1)$.

V. Knichal (Prague).

Mahler, Kurt. Sui determinanti minimi delle sezioni di un corpo convesso. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 251–252 (1948).

Let K be a convex body in the 3-dimensional space, symmetric about the x_3-axis. Let π_z be the plane $x_3=z$ and let K_z be the intersection of K and π_z. Let $\Delta(K_z)$ be the lower bound of the determinants of all plane lattices situated in π_z and containing the point $O_z=[0, 0, z]$ which are K_z-admissible (i.e., which have no point in the interior of K_z except O_z). Theorem: suppose $K(z_1) \neq 0$, $K_{z_2} \neq 0$, $0 \leqq t \leqq 1$, $z=(1-t)z_1+tz_2$; then $\Delta^{\frac{1}{2}}(K_z) \geqq (1-t)\Delta^{\frac{1}{2}}(K_{z_1})+t\Delta^{\frac{1}{2}}(K_{z_2})$. This result follows from an analogous inequality of Brun (where $\Delta(K_z)$ is replaced by the area of K_z) and from a theorem of Reinhardt [Abh. Math. Sem. Hamburg. Univ. **10**, 216–230 (1934)] and Mahler [Nederl. Akad. Wetensch., Proc. **50**, 692–703 = Indagationes Math. **9**, 326–337 (1947); these Rev. **9**, 10] according to which $4\Delta(K_z)$ is the minimum of the areas of all hexagons situated in π_z, containing K_z and symmetric about O_z.

V. Jarník (Prague).

Carlitz, L. Some properties of Hurwitz series. Duke Math. J. **16**, 285–295 (1949).

Consider the class of series $f(x)=\sum_{m=0}^{\infty} c_m x^m/m!$, where the c_m are rational integral (or integral p-adic) coefficients. In this paper general results characterizing such (Hurwitz) series are first derived. Secondly, the particular case when $c_0=0$, $c_1=1$ is studied in detail. Here, the coefficients e_m in the inverse function $\lambda(x)=\sum_1^{\infty} e_m x^m/m!$ are also integral. A typical result is that the assumption $p \mid e_m$ for $m>p$ leads to the congruence $c_{m+p-1} \equiv c_m c_p \pmod p$. Finally, the coefficients $\beta_m^{(k)}$ in $(x/f(x))^k=\sum_{m=0}^{\infty}\beta_m^{(k)}x^m/m!$ $(k \geqq 1)$, are treated; for $f(x)=e^x-1$, $\beta_m^{(k)}$ reduces to the Bernoulli number $B_m^{(k)}$ in Nörlund's notation. Among other results it is shown that $p^k\beta_m^{(k)}$ is integral (mod p), and that $p\beta_m^{(k)}$ is integral for $k=ap^i$, $0<a<p$. The following explicit formula may be cited: $p\beta_m^{(p^r)} \equiv -c_p^n \pmod p$ $(m=np^i(p-1))$; $p\beta_m^{(p^r)} \equiv c_p^n$ $(m=p^{i-1}(p-1)(np+1))$; if m is not of the indicated form then $\beta_m^{(p^r)}$ is integral.

A. L. Whiteman.

Carlitz, L. Congruences for the coefficients of the Jacobi elliptic functions. Duke Math. J. **16**, 297–302 (1949).

In the Jacobi elliptic function $\operatorname{sn} x=\sum_1^{\infty} A_m(k^2)x^m/m!$ $(A_1(k^2)=1)$, where the $A_m(k^2)$ are polynomials in k^2 with integral coefficients, put $a_m=A_m(k^2)$. Let p denote a fixed prime greater than 2 and assume that $l=k^2$ is integral (mod p), that is, the denominator of the rational number l is prime to p. It is first shown that

$$\sum_{i=0}^{r}(-1)^i\binom{r}{i}a_p^{r-i}a_{m+i(p-1)} \equiv 0 \pmod{p^r},$$

where $m \geqq r \geqq 1$. Now put $x/\sin x = \sum_0^\infty \beta_m x^m/m!$ ($\beta_0 = 1$). Then it is shown that $\sum_{i=0}^{r} (-1)^i \binom{r}{i} a_p^{r-1} \tau_{m+i(p-1)} \equiv 0 \pmod{p^r}$, where $\tau_m = \beta_m/m$, $m > r \geqq 1$ and $p-1 \nmid m$. A similar result is also obtained for the case $p-1 \mid m$. These results are analogous to Kummer's congruences for the Euler and Bernoulli numbers, but the standard methods of proof do not carry over since explicit formulas for $A_m(k^2)$ are not available.

A. L. Whiteman (Los Angeles, Calif.).

Carlitz, L. Hurwitz series: Eisenstein criterion. Duke Math. J. 16, 303–308 (1949).

By an H-series of type k, $k \geqq 0$, is meant a formal power series of the form $f(x) = \sum_{m=0}^\infty a_m x^m/\{m, k\}$, where the coefficients a_m are rational integers and

$$\{m, k\} = m!(m+1)! \cdots (m+k)!/1!2! \cdots k!,$$

$\{m, 0\} = m!$. For fixed k, the totality of such series is denoted by $\mathfrak{H}_k$. The order of $f(x)$ is the integer r such that $a_0 = \cdots = a_{r-1} = 0$, $a_r \neq 0$. It is proved that $\mathfrak{H}_k$ is a domain of integrity; that $\mathfrak{H}_0 \subset \mathfrak{H}_k \subset \mathfrak{H}_{k+1}$; and that the quotient of series of type k is expressible in terms of series of type $k+r$. The major result of the paper states that if the series $w(x) = \sum_0^\infty \gamma_m x^m$ with rational coefficients γ_m satisfies the equation $\sum_{i=0}^{t} A_i(x) w^i(x) = 0$, where the coefficients $A_i(x) \in \mathfrak{H}_k$, then there exist integers $c \geqq 1$, $r \geqq k$ such that $cw(cx) \in \mathfrak{H}_r$. The function-theoretic arguments appearing in the usual proof of the Eisenstein criterion are here replaced by a simple lemma. As an application it is shown that for an arbitrary prime p the series $\sum_{m=0}^\infty p^{-m^2} x^m$ is transcendental relative to any $\mathfrak{H}_r$.

A. L. Whiteman.

Rudin, Walter. A theorem on Hurwitz series. Duke Math. J. 16, 309–311 (1949).

[For the definition of an H-series of type k, of the set $\mathfrak{H}_k$ and of the symbol $\{m, k\}$, see the preceding review.] The author proves the transcendence of a class of H-series of type $k+1$ relative to $\mathfrak{H}_k$. As a special case of his theorem, he deduces that $\sum_0^\infty x^m/\{m, k+1\}$ is transcendental relative to $\mathfrak{H}_k$. In particular, considering the simplest case ($k = 0$), he concludes that $\sum_0^\infty x^m/m!(m+1)!$ is transcendental relative to the set $\mathfrak{H}_0$ of series $\sum_0^\infty a_m x^m/m!$.

A. L. Whiteman (Los Angeles, Calif.).

Erdös, P. On arithmetical properties of Lambert series. J. Indian Math. Soc. (N.S.) 12, 63–66 (1948).

Let $f(x) = x(1-x)^{-1} + x^2(1-x^2)^{-1} + x^3(1-x^3)^{-1} + \cdots$ and $g(x) = x(1-x)^{-1} - x^2(1-x^2)^{-1} + x^3(1-x^3)^{-1} - \cdots$. Chowla has proved that $g(1/t)$ is irrational for any integer $t \geqq 5$ [Proc. Nat. Inst. Sci. India 13, 171–173 (1947); these Rev. 9, 500]. In this paper the author extends Chowla's result by showing that both $f(1/t)$ and $g(1/t)$ are irrational for any integer t with $|t| > 1$. To this end he uses the identities $f(x) = \sum_{n=1}^\infty d(n) x^n$ and $4g(x) = \sum_{n=1}^\infty r(n) x^n$, where $d(n)$ and $r(n)$ denote, respectively, the number of divisors of n and the number of representations of n as a sum of two squares. He then shows that the "decimal" representations of $\sum_{n=1}^\infty t^{-n} d(n)$ and $\sum_{n=1}^\infty t^{-n} r(n)$ in the scale of t will contain at least h consecutive zeros, h being an arbitrary positive integer. Since these representations are not finite it follows that these numbers are irrational.

J. Popken (Utrecht).

Erdös, P. Some asymptotic formulas in number theory. J. Indian Math. Soc. (N.S.) 12, 75–78 (1948).

Let $A(n)$ denote the number of integers $m \leqq n$ for which $(m, \varphi(m)) = 1$. The author shows that

$$A(n) = (1+o(1)) \frac{ne^{-\gamma}}{\log \log \log n},$$

where γ is Euler's constant.

W. H. Simons.

Reiner, Irving. A generalization of Meyer's theorem. Trans. Amer. Math. Soc. 65, 170–186 (1949).

A classical theorem asserts that a binary quadratic form with integral coefficients, which represents at least one number of an arithmetic progression, represents an infinity of primes of this progression, provided that the progression contains at least two coprime numbers. The author generalises this theorem; instead of considering the representability of numbers by quadratic forms of discriminant D, he considers a more general division into classes. His formulation, which is too complicated to be given here in detail, could be considerably simplified in the language of quadratic fields. The proof proceeds along conventional lines, but in order to show that his L-series do not vanish at $s = 1$, the author has to assume that $L'(s)$ is bounded for $s > 1$.

H. A. Heilbronn (Bristol).

Zassenhaus, Hans. Über die Existenz von Primzahlen in arithmetischen Progressionen. Comment. Math. Helv. 22, 232–259 (1949).

The purpose of this paper is to give an "elementary" proof of Dirichlet's theorem concerning primes in arithmetical progressions. The author calls a mathematical argument "elementary" when it can be carried out in a finite number of steps without the use of limiting processes. Thus the use of the differential and integral calculus and of functions such as the logarithm is forbidden although complex numbers and the simpler properties of ideals and cyclotomic fields are allowed. The sequence of ideas is very roughly the same as that used in the classical analytical proof, but since they have to be presented in "elementary" form the Dirichlet L-functions have to be replaced by finite sums, and statements concerning continuity and limits by inequalities. Also, an appreciable part of the paper is devoted to the construction and investigation of the properties of an "ersatz" logarithm $l_n(z)$ which is defined for every nonnegative integer n and algebraic numbers z, which are not zero nor negative real numbers, to be $2^n\{z^{2^n} - 1\}$. The establishment of the relevant properties of these finite analogues adds considerably to the complication of the algebra which is, fortunately, carried out in full.

It is not possible to give an adequate summary of this paper here, but it may be stated that the final result depends upon the inequalities

$$\left| \varphi(m) \sum_{p \equiv a (m); p \leqq N} p^{-s} - \sum_{p \leqq N; p \nmid m} p^{-s} \right| \leqq \sum_{\chi \neq \chi_1} \left| \sum_{p \leqq N} \frac{\chi(p)}{p^s} \right| < 6\varphi(m),$$

where p denotes a prime, $\varphi(m)$ is Euler's function, $(a, m) = 1$, χ is any character modulo m and χ_1 the principal character. Since, in "nonelementary" language, the second sum diverges for $s = 1$ as $N \to \infty$, the result will follow if s can be chosen sufficiently near to unity ($s > 1$). The chief difficulty of the proof lies in obtaining a value of s which satisfies the inequalities stated and which is sufficiently close to unity to entail the desired conclusion.

R. A. Rankin.

Selberg, Atle. An elementary proof of Dirichlet's theorem about primes in an arithmetic progression. Ann. of Math. (2) **50**, 297–304 (1949).

The author proves that, for every positive integer k, there exist positive numbers C and x_0 depending only on k such that, when $(k, l)=1$,

$$Q_l(x)=(1/\log x)\sum_{p\leq x;\,p\equiv l(k)} p^{-1}\log p>C$$

(p prime). It follows that every arithmetical progression $ky+l$, where $(k, l)=1$, contains an infinity of primes. The proof does not employ the complex characters modulo k and only finite sums are considered. The formulae

$$\sum_{p\leq x} p^{-1}\log p=\log x+O(1) \quad \text{and} \quad \pi(x)=O(x/\log x)$$

are used; these can be proved by elementary methods. It will therefore be seen that the author's interpretation of the word "elementary" is not so strict as that of Zassenhaus [see the preceding review]. Agreement on interpretation could no doubt be attained without difficulty at the cost of considerable expansion of the algebra. The proof is of interest not only because of its elementary character but because it has so little in common with the classical analytical proof.

By means of the relations $\sum_{d|n}\mu(d)\log^2(x/d)=$ (a) $\log^2 x$ $(n=1)$, (b) $\log p\log(x^2/p)$ $(n=p^\alpha,\ \alpha\geqq 1)$, (c) $2\log p\log q$ $(n=p^\alpha q^\beta,\ \alpha\geqq 1,\ \beta\geqq 1,\ p, q$ different primes), (d) 0 otherwise, which are summed over all $n\equiv l \pmod k$, $n\leqq x$, and by repeated partial summation, it is shown that

$$(1)\quad Q_l(x)\geqq \tfrac{3}{17}\sum_{m_1m_2m_3\equiv l(k)} Q_{m_1}(x^{\frac{1}{3}})Q_{m_2}(x^{\frac{1}{3}})Q_{m_3}(x^{\frac{1}{3}})-O(1/\log x),$$

and

$$(2)\quad Q_l(x)\leqq 2/\varphi(k)+O(\log\log x/\log x).$$

The next step is to show that, for every real nonprincipal character $\chi(n)$ modulo k,

$$(3)\quad \sum_{p\leq x;\,\chi(p)=1} p^{-1}\log p>\tfrac{1}{17}\log x, \qquad x>x_0.$$

The proof makes use of (i) the fact that $\chi(p)$ can be written as the quadratic residue symbol (D/p), where D is an integer which is not a square, and $|D|<k^3$, (ii) the product $P=\prod'|u^2-Dv^2|$ taken over all integers u and v such that $|u|\leqq\{\frac{1}{2}x\}^{\frac{1}{2}}$, $|v|\leqq\{\frac{1}{2}x/|D|\}^{\frac{1}{2}}$, $(u, v)\neq(0, 0)$. The inequalities

$$x|D|^{-\frac{1}{2}}\log x<\log P<8x|D|^{-\frac{1}{2}}\sum_{p\leq x;\,(D/p)=1} p^{-1}\log p+O(x)$$

are derived, the second one being obtained by estimating the number of factors of P which are divisible by each prime power p^α; (3) follows at once.

It can be deduced from (2) and (3) that there is a set S of at least $\frac{1}{2}\varphi(k)$ incongruent residue classes m for which

$$Q_m(x^{\frac{1}{3}})>\tfrac{1}{17}\{\varphi(k)\}^2, \qquad x>x_0,$$

and S contains unity and two members m, m' such that $mm'\equiv l \pmod k$, provided that $Q_l(x)<1/\{30\varphi(k)\}$. It is shown that this implies that S contains three members m_1, m_2, m_3 such that $m_1m_2m_3\equiv l \pmod k$, and (1) then gives

$$Q_l(x)\geqq \tfrac{3}{17}Q_{m_1}(x^{\frac{1}{3}})Q_{m_2}(x^{\frac{1}{3}})Q_{m_3}(x^{\frac{1}{3}})-O(1/\log x)>(20)^{-4}\{\varphi(k)\}^{-4},$$

which proves the theorem. Although the general trend of the argument is clear the paper is not easy to read since the algebraic steps necessary for the estimation of the order of magnitude of sums are mostly omitted. *R. A. Rankin.*

Selberg, Atle. An elementary proof of the prime-number theorem. Ann. of Math. (2) **50**, 305–313 (1949).

Erdös, P. On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. Proc. Nat. Acad. Sci. U. S. A. **35**, 374–384 (1949).

The prime number theorem (PNT) asserts that

$$\pi(x)\sim x/\log x, \qquad x\to\infty,$$

where $\pi(x)$ is the number of primes $p\leqq x$. All previous proofs have been by "transcendental" arguments involving some appeal to the theory of functions of a complex variable. Successive proofs have moderated the demands on this theory, or invoked alternative analytical theories (e.g., Fourier transforms), but there remained a nucleus of complex variable theory, namely the proposition that the Riemann zeta-function $\zeta(s)=\zeta(\sigma+it)$ has no zeros on the line $\sigma=1$; and this could hardly be avoided, except by a radically new approach, since the PNT is in a clearly definable sense "equivalent" to this property of $\zeta(s)$. It has long been recognised that an "elementary" proof of the PNT, not depending on analytical ideas remote from the problem itself, would (if indeed possible) constitute a discovery of the first importance for the logical structure of the theory of the distribution of primes. An elementary (though not easy) proof is given, in various forms, in these two papers. Since the papers are in some ways complementary, it seems clearest to review them together.

The starting point is A. Selberg's formula

$$(1)\quad \sum_{p\leq x}\log^2 p+\sum_{pq\leq x}\log p\log q\sim 2x\log x,$$

where p, q run over primes. The formula might be deduced from the PNT or its equivalent $\vartheta(x)\equiv\sum_{p\leq x}\log p\sim x$, but this, on the classical theory of the distribution of primes, would give it a transcendental basis. The significant events to record are: (a) the discovery by Selberg of an elementary proof (indeed with "$\sim 2x\log x$" replaced by "$=2x\log x+O(x)$," though this refinement is not required for the application); and (b) the recognition that the formula nevertheless embodied facts previously accessible only to the analytical theory of primes. The hope that (1) might therefore be made the basis of an elementary proof of the PNT itself was justified, but the deduction was by no means obvious. The first proof (by Selberg) used, besides (1) (and parts of the elementary theory of primes), the following provisional result (deduced from (1) by Erdös): (2) For any $\lambda>1$ the number of primes in $(x, \lambda x)$ is at least $Kx/\log x$ for $x>x_0$ ($K=K(\lambda)>0$, $x_0=x_0(\lambda)$). Later it was found possible to argue directly from (1) without the intervention of (2).

Selberg's paper contains his proof of (1), a statement of (2), his deduction of the PNT from (1) and (2), and his final direct deduction of the PNT from (1). Erdös's paper contains a statement of (1), an allusion to Selberg's final proof of the PNT (not published at the time), and an account of other proofs in chronological order; the account includes Erdös's own proof of (2), Selberg's deduction of the PNT from (1) and (2), a sketch of Selberg's simplified proof of (2), and the joint simplified direct deduction of the PNT from (1). It is stated (in both papers) that the method can be adapted to the prime number theorem for arithmetical progressions.

Selberg's proof of (1) is, in accordance with the general spirit of the investigation, expressed in arithmetical form. But in a brief summary it is quicker to treat the formula as the result of equating coefficient-sums in the formal

identity $(\zeta'/\zeta)'+(\zeta'/\zeta)^2=\zeta''/\zeta$, where $\zeta=\zeta(s)$, $\zeta'=\zeta'(s)$, $\cdots$. The left hand side gives substantially the left hand side of (1). As to the right hand side we can write

$$\zeta''=2\zeta^3+b\zeta^2+c\zeta+\delta,$$

where b, c are certain constants and δ is a Dirichlet's series $\sum d_n n^{-s}$ with coefficient-sum $D(x)\equiv\sum_{n\leqq x}d_n=O(x^\alpha)$ $(0<\alpha<1)$, by elementary theorems on divisor functions. The coefficient-sum of $\zeta''/\zeta=2\zeta^2+b\zeta+c+\zeta^{-1}\delta$ can now be estimated with an error $O(x)$, and (1) follows. [This is the general idea; but Selberg, besides avoiding even the formal use of Dirichlet's series, arranges the details rather differently so that the coefficients $d_3(n)$ in ζ^3 do not enter.]

Of the various deductions from (1) the joint simplified proof of the PNT is perhaps the easiest to summarise. In this, (1) is used in the equivalent form

$$(1)' \qquad \vartheta(x)\log x+\sum_{p\leqq x}\vartheta(x/p)\log p\sim 2x\log x,$$

which may be rewritten as

$$(1)'' \qquad s(x)+\frac{1}{\log x}\sum_{p\leqq x}\epsilon_p s(x/p)\to 2,$$

where $s(x)=\vartheta(x)/x$ and $\epsilon_p=(\log p)/p$; and we wish to deduce that $\vartheta(x)\sim x$, i.e., $s(x)\to 1$. We have, by elementary theory: (3) $\sum_{p\leqq x}\epsilon_p\sim\log x$; so the second term in (1)'' is essentially a weighted average of $s(x)$. [The situation now recalls Mercer's theorem; a certain linear combination of a function and its average tends to a (finite) limit, and it is required to deduce that the function itself tends to a limit. The proof (by "compensation") has some features in common with Knopp's proof of Mercer's theorem, but it uses special properties of $s(x)$ that have no counterpart in Mercer's theorem, for example the "Tauberian" condition "$xs(x)$ positive and increasing" and the connection of $s(x)$ with the weights ϵ_p.] Suppose that

$$a=\liminf s(x)<\limsup s(x)=A.$$

We have $a>0$ by elementary theory; and it is an easy deduction from (1)'', (3) that: (4) $A+a=2$. Consider a (large) x for which $s(x)$ is large (i.e., near A). From (1)'', (3) we deduce that (to compensate): (α) $s(x/p)$ is small (i.e., near a) for a set X of primes comprising almost all $p\leqq x$ (where "almost all" means "except for a subset over which $\sum\epsilon_p$ is small compared with the same sum taken over the whole set"). Take a particular (small) p_1 in X. Then similarly: (β) $s(x/p_1p)$ is large for a set Y of primes comprising almost all $p\leqq x/p_1$. Next, it follows easily from (1)', (4) and the fact that $\vartheta(\cdot)$ is an increasing function that: (γ) points x', x $(x'>x)$ where $s(\cdot)$ is large and small (or vice versa) must be so far apart that $x'/x>(A/a)-\epsilon$ $(=\lambda$, say) for large x. [This is the sense of lemma 5 of Erdös's paper, but the inequality signs in the conclusion and in most of the proof have become reversed. A list of further misprints (supplied by Erdös) includes: (Lemmas 1, 2) for r read p; (p. 377, line 1) for X read C; (p. 377, line 14 from below) for $\vartheta(x)$ read $\vartheta(x/p)$; (p. 378 (14)) move last square bracket two places to the right; (p. 376, line 9 from below; p. 383, line 3) symbols following $c'>C-$ and $I_4=$ badly printed (but meaning clear).] Now (α), (β), (γ) involve a contradiction. The proof of this, the most difficult step, is presented in alternative forms. The simpler version is based on a study of the sum $S=\sum\epsilon_p\epsilon_q$ taken over all pairs of primes (p, q) for which p is in X, $\lambda^{-1}p<p_1q\leqq\lambda p$. Writing $S=\sum_p\epsilon_p\sum_q\epsilon_q$, we have $\sum_q\geqq(\lambda p/p_1)^{-1}\{\vartheta(\lambda p/p_1)-\vartheta(\lambda^{-1}p/p_1)\}>c>0$ by the definition of a, A, since $a\lambda>A\lambda^{-1}$; whence $S>(c-\epsilon)\log x$, since, by (α), $\sum_p$ extends over almost all $p\leqq x$. On the other hand, writing $S=\sum_q\epsilon_q\sum_p\epsilon_p$, we have

$$\sum_p\leqq(\lambda^{-1}p_1q)^{-1}\vartheta(\lambda p_1q)<C;$$

whence $S<\epsilon\log x$, since, by (γ), q can never belong to Y (because x/p_1q is too near to x/p for some p of X), so that, by (β), $\sum_q$ extends over "almost no" primes. This contradiction shows that $a=A$ $(=1$ since $A+a=2)$. Selberg, in his own final proof, proceeds on somewhat different lines, by "successive approximation." He first derives from (1) an inequality involving (on both sides) the function $R(x)=\vartheta(x)-x$, and then infers from it (and elementary properties of $R(x)$) that, if $|R(x)/x|<\alpha$ (<8) $(x>x_0)$, then $|R(x)/x|<\alpha_1$ $(x>x_1)$, where $\alpha_1=\alpha(1-k\alpha^2)$ and k is a positive absolute constant; whence by repeated application $|R(x)/x|<\alpha_n$ $(x>x_n)$ where $\alpha_n\to 0$ as $n\to\infty$. [The logical arrangement on p. 311 is confusing, since the proof that (3.2) still holds if $R(n)$ changes sign does not seem to be valid until the order of x'/x has been suitably restricted. Also the argument of the text seems to use a sharpened version of (1), though the full form with error $O(x)$ is not required. The claim in the "final remark" that (1) itself suffices would seem to presuppose the use of the alternative form of inequality referred to in footnote 4 on p. 311.]

It may be useful to view the elementary proof against its analytical background, in order to see how the argument, while omitting all reference to analytical facts known to be inseparable from the truth of the PNT, nevertheless avoids the danger of coming into conflict with these facts (and thereby proving too much). Let f be a function of the complex variable $s=\sigma+it$, to be taken as $-\zeta'/\zeta$ in the application but subject at present only to the following restrictions: (i) $f=\sum a_n n^{-s}$ $(\sigma>1$; a_n real); (ii) f is regular in $\sigma\geqq 1$ except possibly for simple poles on $\sigma=1$; (iii) $g\equiv -f'+f^2=\sum b_n n^{-s}$ $(\sigma>1)$, where $B(x)\equiv\sum_{n\leqq x}b_n\sim 2x\log x$. (When $f=-\zeta'/\zeta$, (iii) is equivalent to Selberg's formula (1).) Then, by (iii), g cannot have a double pole at a point $s\neq 1$ of $\sigma=1$; whence any possible pole of f at such a point must have residue R satisfying $R+R^2=0$, i.e., $R=-1$; and the example

$$f(s)=\zeta(s)-\zeta(s-i\alpha)-\zeta(s+i\alpha)$$

$(\alpha>0)$ shows that this is actually possible. Thus our present assumptions are consistent with the existence of poles of f at points $s\neq 1$ on $\sigma=1$; and this may be taken as a "reason" why it is possible to give an elementary proof of (1) without becoming involved in the question of the existence of zeros of ζ on $\sigma=1$. Now introduce the further assumption: (iv) $a_n\geqq 0$. Then it can be proved, as in Hadamard's classical argument, that a pole of f at $s=1+it$ with $R=-1$ implies a pole at $s=1+2it$ with $R=+1$, and this is impossible; so f can have no pole on $\sigma=1$ except at $s=1$. Thus the properties (i)–(iv) of $f=-\zeta'/\zeta$ do embody the essential analytical fact on which previous proofs of the PNT have been based. What Selberg and Erdös do is to deduce the PNT directly from the arithmetical counterparts of (i), (iii), (iv), without the explicit intervention of the analytical fact. In principle this opens up the possibility of a new approach, in which the old logical arrangement is reversed and analytical properties of $\zeta(s)$ are deduced from arithmetical properties of the sequence of primes. How far the practical effects of this revolution of ideas will penetrate into the structure of the subject, and how much of the theory will ultimately have to be rewritten, it is too early to say.

A. E. Ingham (Cambridge, England).

*van der Corput, J. G. **Démonstration élémentaire du théorème sur la distribution des nombres premiers.** Math. Centrum Amsterdam, Scriptum no. 1, 32 pp. (1948).

This mimeographed pamphlet is a connected account of the elementary proof of the prime number described in the preceding review. Based on lectures by Erdös, it follows the general lines of the joint simplified proof of Selberg and Erdös, but it has been cast into its present form by van der Corput. The author has adopted a very elementary standpoint, so that the text includes proofs of results that would normally be quoted from elementary analysis. The most substantial change is in the concluding stages of the argument; here the result (γ) [see the preceding review] is not used explicitly, but the double sum S is replaced by a triple sum $\sum\epsilon_p\epsilon_q\epsilon_r$. The exposition is clear and precise, and it is only in these final stages, which remain the most difficult part of the proof, that the reader is seriously taxed.

A. E. Ingham (Cambridge, England).

Ramaswami, V. The number of positive integers $\leqq x$ and free of prime divisors $>x^c$, and a problem of S. S. Pillai. Duke Math. J. 16, 99–109 (1949).

Let $f(x, c)$ denote the number of integers specified in the title. The main results of the paper are the formulas

$$f(x, c)=x\varphi(c)+q(c)(x/\log x)+O(x^c/\log x)+O(x/\log^2 x), \qquad \tfrac{1}{2}\leqq c<1;$$

$$f(x, c)=x\varphi(c)+q(x)(x/\log x)+O(x/(\log x)^{\frac{1}{2}}), \qquad c<\tfrac{1}{2} \text{ or } c\geqq 1.$$

Here $\varphi(c)$ is bounded and positive for $c>0$ and $q(c)$ is bounded, positive for $0<c<1$, and zero elsewhere. Explicit formulas are given for the functions $\varphi(c)$ and $q(c)$ and it is shown that they have derivatives of all orders except at $c=1/n$, where n is a positive integer. The functions also satisfy certain inequalities of which $\Gamma(1/c)\varphi(c)<Ac$ for $c>0$ is an example. The problem of Pillai referred to in the title is that of answering the question: is $f(x, c)<Ax^c$ for $c\geqq(\log 2)/(\log x)$? It follows from the formulas of this paper and the inequalities satisfied by $\varphi(c)$ that the answer is "yes."

The proof of the main results uses the prime number theorem in the form $\pi(x)=\int_2^x(\log x)^{-1}dx+O(x/\log^3 x)$. The first result quoted above follows from this by noting that

$$f(x, 1)-f(x, c)=\sum_{n=1}^{r}\pi(x/n)-r\pi(x^c),$$

where $r=[x^{1-c}]$ and $\frac{1}{2}\leqq c<1$. In a note at the end of the paper, the author states a generalization of his results.

R. D. James (Vancouver, B. C.).

Wintner, Aurel. A note on Mertens' hypothesis. Revista Ci., Lima 50, 181–184 (1948).

The author notes that the statements

$$(1)\quad \sum_{n=1}^{x}\mu(n)=O(x^{\frac{1}{2}}), \qquad (2)\quad \sum_{n=1}^{x}\mu(n)/n^{\frac{1}{2}}=O(1),$$

of which (1) is a famous (but doubtful) hypothesis of Stieltjes and Mertens, stand in the following logical relation. By partial summation, (2)$\rightarrow$(1), but not conversely. However, by a theorem of M. Riesz [Acta Math. 40, 349–361 (1916), p. 350, footnote 4] (1)(3)$\rightarrow$(2), where (3) is the statement: $1/\zeta(s)$ is regular at $s=\frac{1}{2}$. Since (3) is true, (1) and (2) are thus both true or both false. *A. E. Ingham.*

Hua, Loo-Keng. An improvement of Vinogradov's mean-value theorem and several applications. Quart. J. Math., Oxford Ser. 20, 48–61 (1949).

This paper gives a self-contained and clear treatment of the things mentioned in the title. The so-called mean-value theorem states that the number of solutions of the simultaneous Diophantine system of equations $\sum_{i=1}^{s}(x_i^h-y_i^h)=0$ $(1\leqq h\leqq k)$, subject to $T<x_i,\ y_i\leqq T+P$, does not exceed $(7s)^{4ki}(\log P)^{2i}P^{2s-k(k+1)/2+\delta}$ provided $s\geqq k(k+1)/4+lk$, $P\geqq 2$ and $\delta=k(k+1)(1-1/k)^l/2$. (Throughout the paper $k>1$.) This is a considerable improvement over the result in the author's book [reviewed below] which essentially required that s be larger by a factor of about $2k$. The proof is somewhat similar to that given in the book.

From this result is obtained the following estimate of Weyl's sum $S=\sum_{x=1}^{P}\exp(2\pi i\alpha x^k)$. Let $|\alpha-h/q|\leqq 1/q^2$, $(h, q)=1$ and $P\leqq q\leqq P^{k-1}$. Then $|S|<c(k)P^{1-1/\sigma}$, where σ is asymptotic to $4k^2\log k$ as $k\rightarrow\infty$. A value of σ asymptotic to $6k^2\log k$ is given in Vinogradov's book [see the second following review] and a value asymptotic to $k^2\log k$ is given in Hua's book. The author mentions that this result can be used to show that the Hardy-Littlewood asymptotic formula for the number of representations of N as the sum of s kth powers holds if $s>s_0\sim 4k^2\log k$. This compares with $s>s_1\sim 10k^2\log k$ as given in Vinogradov's book. Thus, the results of this paper and Vinogradov's book are of the same depth; the author remarks that Vinogradov's method seems to have reached a final stage.

A number of obvious misprints were noted. In addition, the following two rather minor corrections, in no way affecting the validity of the main results, may be mentioned: in theorem 4, the estimate for S should have an additional factor of $\log P$, and in theorem 3 the estimate for the integral should have an additional factor of 2^k.

L. Schoenfeld (Urbana, Ill.).

*Hua, Loo-Keng. **The additive prime number theory.** Trav. Inst. Math. Stekloff 22, 179 pp. (1947). (Russian. English summary)

This is a Russian translation by B. I. Segal and D. A. Vasil'kov of an English manuscript written in 1941. It gives a detailed exposition, starting from scratch, of what was then known on the problem of representing positive integers as sums of a given number of kth powers of primes (the so-called Waring-Goldbach problem). The attack on this problem was begun by Vinogradov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 16, 131–132 (1937); Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 3, 1–67 (1938)] and continued by the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 17, 167–168 (1937); 18, 4 (1938); Math. Z. 44, 335–346 (1938)]. However, much of the material in the present memoir is original work. The method used in all this work is an extension of that used by Vinogradov in treating the special case of the Goldbach problem [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 291–294 (1937); Rec. Math. [Mat. Sbornik] N.S. 2(44), 179–194 (1937)].

In the first chapter the author proves the following two theorems on trigonometrical sums [cf. Hua, J. Chinese Math. Soc. 2, 301–312 (1940); these Rev. 2, 347]. If $f(x)=a_kx^k+\cdots+a_1x$, where $a_1, \cdots, a_k$ are integers such that $(a_1, \cdots, a_k, q)=1$, then

$$\left|\sum_{x=1}^{q}\exp(2\pi i f(x)/q)\right|\leqq c(k, \epsilon)q^{1-1/k+\epsilon},$$

where ϵ is an arbitrary positive number and $c(k, \epsilon)$ denotes a positive constant depending only on k and ϵ. If further $(a_2, \cdots, a_k, q)=1$, then

$$\left|\sum_{x=1}^{m}\exp(2\pi i f(x)/q)-mq^{-1}\sum_{x=1}^{q}\exp(2\pi i f(x)/q)\right|\leqq c_1(k,\epsilon)q^{1-1/k+\epsilon}$$

for any positive integer m. It is to be noted that both in the original paper and in this book the additional hypothesis is not introduced in the second theorem, but unless this is done the estimate must be multiplied by $(a_2, \cdots, a_k, q)^{1/k}$.

In chapter II the results of the first chapter are used to prove the following theorem on the divisors of the values of a polynomial function. If $f(x_1, \cdots, x_n)$ is a polynomial of kth degree with integral coefficients having greatest common divisor unity, then

$$\sum_{x_1=1}^{P}\cdots\sum_{\substack{x_n=1\\ f(x_1,\cdots,x_n)\neq 0}}^{P} d^l(|f(x_1,\cdots,x_n)|)\leqq c_2(k,n,l)A(\log X)^{\sigma(k,n,l)},$$

where $d(m)$ denotes the number of divisors of m, X is the maximum value of $|f(x_1, \cdots, x_n)|$ for $1\leqq x_1, \cdots, x_n\leqq P$, and $A=\max(P^n, X^{n/k})$.

The preceding result is applied in chapter III to prove the following theorem on the mean value of a trigonometrical sum. Let $f(x)$ be an integral-valued polynomial of the kth degree and put $T(\alpha)=\sum_{x=1}^{P}\exp(2\pi i f(x)\alpha)$; then for $1\leqq\nu\leqq k$ we have

$$\int_0^1|T(\alpha)|^{\nu}d\alpha\leqq c_4(k,\nu)P^{\nu-\nu}(\log P)^{\sigma(k,\nu)}d^{\nu-1}(u),$$

where u is the greatest common divisor of the numerators of the coefficients of $f(x)$.

Chapter IV is essentially the same as a paper of the author [Quart. J. Math., Oxford Ser. 11, 161–176 (1940); these Rev. 2, 150] proving the following theorem. If $2\leqq k\leqq 10$ and $C_k=C_k(\alpha_1, \cdots, \alpha_k)=\sum_{x=1}^{P}\exp(2\pi i\{\alpha_k x^k+\cdots+\alpha_1 x\})$, then

$$\int_0^1\cdots\int_0^1|C_k|^{\lambda}d\alpha_1\cdots d\alpha_k\leqq c_5(k,\epsilon)P^{\lambda-\frac{1}{2}k(k+1)+\epsilon},$$

where $\lambda=\lambda(k)$ is an even integer which is tabulated: $\lambda(2)=6, \cdots, \lambda(10)=9190$. This result is a sharpening, for small k, of the so-called Vinogradov mean value theorem [cf. Vinogradov, Rec. Math. [Mat. Sbornik] N.S. 3(45), 435–470 (1938), lemma 6].

In chapter V is given an original proof of the Vinogradov mean value theorem itself, the statement of which is as follows. Suppose $C_k=\sum_{x=1}^{P}\exp(2\pi i\{\alpha_k x^k+\cdots+\alpha_1 x\})$, $b=b(k)=2[\frac{1}{4}(k+1)(k+2)]$, and $k\leqq n\leqq c_7(k)$; then

$$\int_0^1\cdots\int_0^1|C_k|^{bn}d\alpha_1\cdots d\alpha_k\leqq c_8(k)P^{bn-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)\sigma},$$

where $\sigma=(1-1/k)^n$. This mean value theorem has (among others) the following two consequences. If $f(x)$ is an integral-valued polynomial of degree k, then

$$\int_0^1\left|\sum_{x=1}^{P}\exp(2\pi i f(x)\alpha)\right|^{bn}d\alpha\leqq c_9(k)P^{bn-\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)\sigma},$$

where $b=2[\frac{1}{4}(k+1)(k+2)]$, $k\leqq n\leqq c_{10}(k)$, $\sigma=(1-1/k)^n$. If $\alpha_1, \cdots, \alpha_k$ are real numbers, $k\geqq 14$, and $|\alpha_k-h/q|\leqq q^{-2}$, where h and q are coprime integers, $q>0$, then for $P^{k-\epsilon}\leqq q\leqq P^{k-1+\epsilon}$ we have

$$\left|\sum_{x=1}^{P}\exp(2\pi i\{\alpha_k x^k+\cdots+\alpha_1 x\})\right|\leqq c_{11}(k,\epsilon)P^{1-\rho+\epsilon},$$

where $\rho>2/\{k^2(\log k^2+2.2\log\log k^2)\}$. An improved version of this chapter is reviewed above.

A theorem of Vinogradov [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Math. Inst.] 3, 1–67 (1938), Satz 1] which is basic for the Waring-Goldbach problem is given in chapter VI. However, it is modified here so as to be applicable also to the simultaneous Waring-Goldbach problem. Suppose $0<Q\leqq c_{12}(k)(\log P)^{\sigma_1}$, $\alpha_1, \cdots, \alpha_{k-1}$ are real numbers, h and q are coprime integers, and $(\log P)^{\sigma}<q\leqq P^k(\log P)^{-\sigma}$; then if σ_0 is a given positive number and $\sigma\geqq 2^{2k}(\sigma_0+\sigma_1+1)$, we have

$$\left|\sum_{p\leqq P;\,p\equiv l(\mathrm{mod}\,Q)}\exp(2\pi i\{hq^{-1}x^k+\alpha_{k-1}x^{k-1}+\cdots+\alpha_1 x\})\right|\leqq c_{13}(k)P(\log P)^{-\sigma_0}Q^{-1}.$$

Using the results of the previous chapters and the Hardy-Littlewood-Vinogradov method, the author in chapter VII establishes an asymptotic formula for the number of solutions of the Waring-Goldbach problem. More generally, he proves the following result. Let $f(x)$ be an integral-valued polynomial of kth degree with positive first coefficient A; suppose there does not exist an integer q such that $f(x)\equiv f(0)$ (mod q) for all integers x; let $I_s(N)$ denote the number of solutions of the equation $f(p_1)+\cdots+f(p_s)=N$, where the p's are primes; then for $s\geqq 2^k+1$ if $1\leqq k<14$ and $s\geqq k^2(\log k+2.2\log\log k)$ if $k\geqq 14$, we have

$$\left|I_s(N)-A^{-s}\frac{\Gamma^s(1/k)}{\Gamma(s/k)}\frac{N^{s/k-1}}{(\log N)^s}\mathfrak{S}(N)\right|<c_{14}(k,s,f)\frac{N^{s/k-1}\log\log N}{(\log N)^{s+1}},$$

where $\mathfrak{S}(N)$ is the so-called singular series (which can of course be given explicitly). This is the only point in the memoir at which the treatment is not self-contained, for use is made of a theorem on the number of primes in an arithmetic progression when the difference of the progression grows slowly with the increasing of the number of terms of the progression [cf. Walfisz, Math. Z. 40, 592–607 (1935), Hilfssatz 3]. This theorem in turn is a consequence of the Siegel lower estimate of the value at $s=1$ of Dirichlet's $L(s, \chi)$ functions with real characters χ [Acta Arith. 1, 83–86 (1935)]. However, a simple proof of Siegel's theorem has been given recently by Estermann [J. London Math. Soc. 23, 275–279 (1948); these Rev. 10, 356], so that the lemma of Walfisz is not as deep as was formerly supposed.

The singular series for the Waring-Goldbach problem is studied in chapter VIII [cf. Hua, Math. Z. 44, 335–346 (1938)] and the following result is proved. Suppose $K=2$ for k odd and $K=2\prod_{(p-1)|k}p^{\theta+1}$ for k even, where $\theta=\theta(p)$ is the exact power to which p divides k; then for $s\geqq 3k+1$ and $N\equiv s$ (mod K) the singular series $\mathfrak{S}(N)$ for $f(x)=x^k$ is greater than a positive constant. In combination with the theorem of the previous chapter this result shows that, if $H(k)$ denotes the smallest natural number s such that every sufficiently large integer $N\equiv s$ (mod K) is a sum of s kth powers of primes, then $H(k)\leqq 2^k+1$ if $1\leqq k<14$ and $H(k)\leqq k^2(\log k+2.2\log\log k)$ if $k\geqq 14$.

In chapter IX the author improves the result just mentioned to the following: $H(k)\leqq 2k+2m+7\sim 6k\log k$, where $b=k^2(\log k+1.1\log\log k^2)$ if $k\geqq 14$, $b=2^{k-1}$ if $4\leqq k<14$, and $m=[\{\log\frac{1}{2}b+\log(1-2/k)\}/\{-\log(1-1/k)\}]$. He also proves that $H(4)\leqq 15$, $H(5)\leqq 25$ and states that $H(6)\leqq 39$, $H(7)\leqq 55$, $H(8)\leqq 75$.

The next two chapters treat the simultaneous Waring-Goldbach problem, i.e., the problem of finding sufficient conditions for the solvability of the system of Diophantine equations $p_1^k+\cdots+p_s^k=N_k, \cdots, p_1+\cdots+p_s=N_1$, where the p's are primes. In chapter X an asymptotic formula for the number of solutions is established for $s\geqq s_0\sim 4.14k^2\log k$. In chapter XI the question of the solvability of the system is discussed for $s\geqq s_0\sim 7k^2\log k$.

In chapter XII other problems are mentioned which may be attacked by the method of this memoir. Finally in chapter XIII a corollary of the Vinogradov mean value theorem is derived and some of its important consequences are stated.

Since the writing of this work considerable progress has been made in the Waring-Goldbach problem [see, for example, Vinogradov, C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 182–183 (1942); these Rev. **4**, 211]. In particular, with reference to chapter V, the author's paper reviewed above shows that the asymptotic formula in the Waring-Goldbach problem can be established for $s\geqq s_0\sim 4k^2\log k$ and that $H(k)\leqq s_0\sim 4k\log k$; also in the simultaneous problem the asymptotic formula can be established for $s\geqq s_0\sim 6k^2\log k$ and the solvability can be discussed for $s\geqq s_0\sim 3k^2\log k$. In addition one must notice the new proofs by Linnik [Rec. Math. [Mat. Sbornik] N.S. **19**(61), 3–8 (1946); these Rev. **8**, 317] and Tchudakoff [Ann. of Math. (2) **48**, 515–545 (1947); these Rev. **9**, 11] of the Goldbach-Vinogradov theorem; these proofs use rather deep theory of the $L(s,\chi)$ functions instead of the Vinogradov estimations of trigonometrical sums. *P. T. Bateman.*

* **Vinogradov, I. M. The method of trigonometrical sums in the theory of numbers.** Trav. Inst. Math. Stekloff **23**, 109 pp. (1947). (Russian)

The present book is largely a revision of the author's earlier work [same Trav. **10** (1937)]. However, a number of sharper results are obtained in the present work. Chapter I [16 pp.] deals with general lemmas which are required for the development of the succeeding chapters. In chapter II [8 pp.], the singular series for Waring's problem is studied. In chapter III [4 pp.], there is obtained an asymptotic formula for an integral connected with Waring's problem. In chapter IV [5 pp.], there is obtained an estimate which implies the slightly weaker $G(n)<n(3\log n+11)$ for $n>2$. The best previous result, for large n, was $G(n)<n(4\log n+8\log\log n+12)$ [Vinogradov, Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **5**, 153–180 (1938)]. Since $G(n)>n$, there is still a gap in our knowledge of the behavior of $G(n)$. Chapter V [6 pp.] deals with approximation by fractional parts of polynomials. The result is the same as that given in 1937.

In chapter VI [15 pp.], an advance is made in the estimation of the Weyl sum $S=\sum_{x=1}^{P}\exp\{2\pi i m F(x)\}$, where m and P are integers, $F(x)=a_{n+1}x^{n+1}+\cdots+a_1x$, and $a_{n+1}, \cdots, a_1$ are real. As is well known, sums of this type are basic in many problems of number theory; e.g., estimations of S are needed for estimating $\zeta(\sigma+it)$ in the neighborhood of $\sigma=1$, for estimating the error term in $\pi(x)\sim\operatorname{li} x$, determining the range of convergence of the singular series for Waring's problem, etc. This result, simplified in accordance with a footnote of the paper reviewed below, is: let $n>10$, let s be one of the integers $2, \cdots, n+1$, $|a_s-a/q|\leqq 1/q^2$, $(a,q)=1$, let τ be a constant such that $0<\tau\leqq 1$ and $0<c_1P^\tau\leqq q\leqq c_2P^{s-\tau}$; then there exists an absolute constant $c(n)$ such that $|S|<c(n)P^{1-\rho t}m^{\rho_0/\tau}$, where $t=1+(30n)^{-1}$ and $\rho^{-1}=(3n^2/\tau)\log\{12n(n+1)/\tau\}$. The previous result differed from this, in its most essential part, by having n^3 in place of n^2 in the expression for ρ^{-1}. Consequently the gain is quite substantial. Two other results, estimating sums of the form $\sum_{x=N+1}^{N+P}\phi(x)\exp\{2\pi i f(x)\}$ for monotonic $\phi(x)$ and real-valued $f(x)$ whose $(n+1)$th derivative satisfies certain order conditions, are also given.

In chapter VII [4 pp.], the Hardy-Littlewood asymptotic formula for the number of representations of N as the sum of r positive nth powers (Waring's problem) is shown to be valid for $r\geqq[10n^2\log n]$ if $n>11$. This, too, improves Vinogradov's earlier result by a factor of about n. Chapter VIII [3 pp.] deals with the distribution of the fractional parts of polynomials where a similar improvement is obtained. In chapter IX [18 pp.] are given estimates of trigonometric (i.e., exponential) sums in which the summation variable runs over the prime numbers in a certain interval. The Goldbach-Vinogradov theorem, that every sufficiently large odd number is the sum of three primes, is proved in chapter X [9 pp.]. The proof takes for granted the work of Page and Siegel on the error term in the asymptotic formula $\pi(N,q,l)\sim(\operatorname{li} N)/\varphi(q)$. In chapter XI [2 pp.] a result is given on the distribution of the fractional parts of the function αp, where p is a prime.

In addition to the errata listed in the book, nineteen others were noted. The least obvious of these are: on page 48, third line, $2n+1$ should be replaced by $4n$; on page 73, five lines from the bottom, $2\pi i f(x)$ should be replaced by $2\pi i\alpha f(x)$; page 102, line 2, $\sqrt{2}$ should be $\sqrt{r}$; the correct title for reference no. 17 in the bibliography is "On Waring's problem."

This book is a worthwhile addition to the too few books on analytic number theory and to the practically nonexistent expositions of the work on exponential sums of the Russian school. Together with the recent book of Hua in the same series [see the preceding review], there is now made available a connected treatment of a substantial part of the techniques and results which have been obtained since Landau's Vorlesungen and Fortschritte.

L. Schoenfeld (Urbana, Ill.).

Vinogradov, I. M. On an estimate of trigonometric sums with prime numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 225–248 (1948). (Russian)

The main theorems in this paper are the analogues of those of chapters VI and VIII of the author's book reviewed above. The main result is the following. Let $0<\kappa\leqq 1$, $l>0$ and integral, $n>9$, p a prime, $f(p)=a_np^n+\cdots+a_1p$, where $a_n, \cdots, a_1$ are real, and for a certain s, which is one of the integers $2, \cdots, n$, we have $|a_s-a/q|\leqq 1/q\tau$, $(a,q)=1$, $d>0$ a constant, $dP^s\leqq q\leqq\tau=P^{s/2}$; further, let $\rho=.041/n^2(2+\log n)$ and $\rho_0=(.416/.041)\rho$ if $q>P^{\frac{1}{2}}$, while $\rho=.37\kappa/n^2(4+\log n^2/\kappa)$ and $\rho_0=(.375/.37)\rho$ if $q\leqq P^{\frac{1}{2}}$. If $l\leqq P^{\rho_0\kappa}$ we then have $|\sum_{p\leq P}\exp\{2\pi i l f(p)\}|<kP^{1-\rho}$ for some $k(n)$. Similar results are obtained when $f(p)$ is not a polynomial and when p has the range $P_1<p<P_2$. Finally, the estimate $\gamma\{\pi(P_2)-\pi(P_1)\}+O(P^{1-\rho'})$ is given for the number of primes p such that $P_1<p<P_2$ and $0\leqq f(p)-[f(p)]<\gamma$; here $\rho'=.044/n^2(\log n+2)$. *L. Schoenfeld* (Urbana, Ill.).

Vinogradov, I. M. On the distribution of products of prime numbers and the numerical function of Möbius. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 341–350 (1948). (Russian)

This paper contains proofs of results previously announced without proof [C. R. (Doklady) Acad. Sci. URSS

(N.S.) 30, 681–682 (1941); these Rev. 3, 165]. The main theorem is as follows. Let m, a, q be integers, $m>0$, $q>0$, $(a,q)=1$; let $\epsilon_0>0$, $\tau=\log N$, $|\theta|\leqq 1$, $N^{\frac{1}{2}}\leqq\tau\leqq N\exp(-\tau^{\epsilon_0})$, $\alpha=a/q+\theta/q\tau$, $\exp(\tau^{\epsilon_0})\leqq q\leqq\tau$, $\Delta=(1/q+q/N)^{\frac{1}{2}}$, $f=\Delta^{-1}$. Then, for $K<f^2$,

$$\sum_{k=1}^{K}\left|\sum_{w\leqq N}e^{2\pi i\alpha kw}\right|\ll KN(\Delta^{1-\epsilon}+N^{-0.3+\epsilon}),$$

where w runs through all numbers which are products of m distinct prime factors. This theorem is a generalization of theorem 3 in chapter 9 of the author's book [see the second preceding review] and the proof is analogous to the proof in the book. Applications are given to sums of the type $\sum_{n=1}^{N}\mu(n)$, where $\mu(n)$ denotes the Möbius function. A typical theorem is the following. Let n, q be positive integers and n' the least nonnegative residue of n mod q. Let $0<\beta\leqq 1$, $\tau=\log N$, $\exp(\tau^{\epsilon_0})\leqq q\leqq N\exp(-\tau^{\epsilon_0})$. Put $M(\beta)=\sum\mu(n)$, where n runs through all positive integers n not exceeding N such that $n'<\beta q$. Then

$$M(\beta)=\beta M(1)+O(N\Delta_2),\quad \Delta_2=(1/q+q/N)^{0.5-\epsilon}+N^{-0.3+\epsilon}.$$

The author states that "further attempts to refine these results by means of my method have met with considerable difficulty." *A. L. Whiteman* (Los Angeles, Calif.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Chittenden, E. W. On the general theory of functions. Proc. Iowa Acad. Sci. 54, 207–210 (1947).

This paper contains a brief treatment of functions considered as classes of ordered pairs and also of set-functions. Some properties of functions and set-functions are developed from this point of view. *I. L. Novak.*

Loomis, Lynn H. Haar measure in uniform structures. Duke Math. J. 16, 193–208 (1949).

Let S be a uniform structure such that the smallest number of x-neighborhoods required to cover a y-neighborhood is finite and the same for all y-neighborhoods. Assume also that the neighborhoods are symmetric and the indices densely ordered. In terms of the Haar covering functions a class Z of subsets of S is distinguished on which the Haar ratios converge to a finitely additive limit m. Then Z is identified with the field of "zero-boundary" sets, and m with the Jordan content, corresponding to the Haar measure μ determined by m. The term measure is here used to denote a finite nonnegative function defined for totally bounded open sets, monotone increasing, additive for sets having disjoint neighborhoods, and such that $\mu(O_1\cup O_2)\leqq\mu(x[O_1])+\mu(x[O_2])$ for every index x. In case S is locally compact the Jordan content corresponding to such a measure admits a countably additive Borel extension. The invariance and uniqueness of Haar measure are established, allowing for certain exceptional indices. A correspondence is set up between measures in S and in its locally compact completion T which permits Haar measure in S to be regarded as the relativization of Haar measure in T. As an application it is shown that if G is a group of homeomorphisms of a topological space T such that (1) the images of any nonvoid open set cover T, (2) there is a nonvoid open set which can be covered by a finite number of images of any nonvoid open set, and (3) if O is an open neighborhood of p there is a nonvoid open set of which every image that contains p is a subset of O, then T admits a unique measure invariant under G. *J. C. Oxtoby.*

Banach, S. On measures in independent fields. Studia Math. 10, 159–177 (1948).

A collection $\{A_i\}$ of Boolean σ-algebras of subsets of a set is countably independent if any countable class of sets, any two distinct ones of which belong to different algebras of the collection, has a nonempty intersection. The principal theorem of the paper asserts that if, for each i, μ_i is a countably additive finite ~~measure~~ probability on A_i, then there exists a measure μ on the least Boolean σ-algebra containing all A_i, such that μ is an extension of each μ_i and such that any countable class of sets, any two distinct ones of which belong to different algebras of the collection, is stochastically independent. *P. R. Halmos* (Chicago, Ill.).

Helson, Henry. Remark on measures in almost-independent fields. Studia Math. 10, 182–183 (1948).

An example is given to answer a problem proposed by Marczewski [Colloquium Math. 1, 122–132 (1948), p. 130; these Rev. 10, 23] on the extension of measure functions. *J. L. Doob* (Ithaca, N. Y.).

Pettineo, Benedetto. Sulla convergenza puntuale delle successioni di insiemi di funzioni quasi continue. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 6, 478–503 (1947).

The author uses the terminology of M. Picone [Teoria moderna dell'integrazione delle funzioni, Pisa, 1945] and, moreover, applies and generalizes some notions discussed by himself in previous papers [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 301–306, 916–918 (1946); these Rev. 8, 449, 450]. Let $\alpha(T)$ be a determining function (i.e., additive and of bounded variation) of the linear intervals T and let F be an infinite set of point functions $f(P)$ quasi-continuous with respect to $\alpha(T)$ in a linear interval Δ. The author calls $\varphi(P)$ a "function of approximate accumulation" of F if to every $\epsilon>0$ there correspond infinitely many functions $f(P)$ of F, such that $|f(P)-\varphi(P)|\leqq\epsilon$ in every point P of Δ, except perhaps for the points of a set X of measure (with respect to $\alpha(T)$) smaller than ϵ. Moreover, let $\{F_n\}$ be a sequence of sets of quasi-continuous functions $f(P)$ in Δ with the properties: (1) $F_n\supset F_{n+1}$, (2) to every $\epsilon>0$ there correspond a set F_n and a number $\delta>0$ such that for every function $f(P)$ of F_n one has $|f(P)-f(P')|\leqq\epsilon$ for the pairs of points P, P' of Δ with distance $PP'\leqq\delta$, except perhaps for the points of a set X of measure smaller than ϵ. In this case he calls $\{F_n\}$ a "sequence of sets of quasi-similarly continuous functions." Then he proves the following theorems. In order that a set F of functions $f(P)$, quasi-continuous and uniformly bounded in the linear interval Δ, have a function of approximate accumulation in Δ, it is necessary and sufficient that there exists a sequence of infinite sets of quasi-similarly continuous functions of F. Moreover, every function of approximate accumulation of a set of quasi-continuous functions is also quasi-continuous. The author applies his results to establish the following generalization of a theorem of G. Fichera [cf. M. Picone, loc. cit., p. 149]: if $f(P)$ is an absolutely integrable function in a linear set Δ, then one

can construct a measurable set L, contained in Δ, on which $f(P)$ is quasi-continuous and such that $\int_L f(P)d\alpha = \int_\Delta f(P)d\alpha$.

A. Rosenthal (Lafayette, Ind.).

Morse, Marston, and Transue, William. Functionals of bounded Fréchet variation. Canadian J. Math. 1, 153–165 (1949).

In the sequel, $k(s, t)$ denotes a real-valued function, defined on the rectangle $Q: a \leqq s \leqq a'$, $b \leqq t \leqq b'$, satisfying the conditions $k(a, t)=0$, $k(s, b)=0$. The Fréchet variation $P(k)$ of k over Q is defined as follows. Take subdivisions $a=s_0 \leqq s_1 \leqq \cdots \leqq s_m=a'$, $b=t_0 \leqq t_1 \leqq \cdots \leqq t_p=b'$ of the intervals $[a, a']$, $[b, b']$ respectively. Choose numbers $e_1, \cdots, e_m$, $e_1', \cdots, e_p'$, equal to ± 1. Set

$$\Delta_{ij}(k) = k(s_{i-1}, t_{j-1}) - k(s_i, t_{j-1}) + k(s_i, t_j) - k(s_{i-1}, t_j).$$

Then $P(k)$ is the least upper bound of all the sums $\sum\sum e_i e_j' \Delta_{ij}(k)$. Assume that $P(k) < \infty$. Then a series of theorems hold which are analogous to theorems on functions of bounded variation of a single variable. Samples follow. (i) The following limits exist everywhere in Q: $k^2(s, t) = \lim k(s, t+v)$, $v \downarrow 0$; $k^{-2}(s, t) = \lim k(s, t+v)$, $v \uparrow 0$; $k^1(s, t) = \lim k(s+u, t)$, $u \downarrow 0$; $k^{-1}(s, t) = \lim k(s+u, t)$, $u \uparrow 0$; (ii) $P(k^m) \leqq P(k)$, $m = \pm 1, \pm 2$. (iii) As a consequence of (i), (ii), the limits $k^{m,n} = (k^m)^n$, $m, n = +1, \pm 2$, exist in Q. (iv) In the (u, v)-plane, denote by $H_{1,1}$, $H_{-1,1}$, $H_{-1,-1}$, $H_{1,-1}$ the quadrants $(u>0, v>0)$, $(u<0, v>0)$, $(u<0, v<0)$, $(u>0, v<0)$, respectively. Then for each interior point of Q, and for $m, n = \pm 1, \pm 2$, $|m| \neq |n|$, the limit $k^{(m,n)}(s, t) = \lim k(s+u, v+t)$, $(u, v) \to (0, 0)$, $(u, v) \varepsilon H_{m,n}$, exists. Furthermore, $k^{(m,n)} = k^{m,n} = k^{n,m}$, $P(k^{(m,n)}) \leqq P(k)$ and $P(k^{(m,n)})$ is independent of the choice of m, n (subject to the conditions $m, n = \pm 1, \pm 2$, $|m| \neq |n|$). (v) The points of discontinuity of k can be covered by a countable set of lines parallel to the s and t axes, respectively. The authors state that these results have applications which will be developed in subsequent papers.

T. Radó.

Mambriani, A. Sul concetto del Tonelli di funzione, di due variabili, a variazione limitata. Atti Sem. Mat. Fis. Univ. Modena 1, 121–130 (1947).

Let $f(x, y)$ be a real-valued function defined on the rectangle $R: a \leqq x \leqq a'$, $b \leqq y \leqq b'$. Let $a=x_0<x_1<\cdots<x_m=a'$ be a subdivision of the interval (a, a'). The author denotes by $V_R^x(f)$ the least upper bound of all the sums of the form $\sum \int_b^{b'} |f(x_i, y) - f(x_{i-1}, y)| dy$ (where the integrals, in the Lebesgue sense, are assumed to be meaningful, perhaps equal to $+\infty$). The quantity $V_R^y(f)$ is defined similarly. The following theorem is then established. For a continuous f, the condition $V_R^x(f) + V_R^y(f) < \infty$ is equivalent to bounded variation in the Tonelli sense.

T. Radó.

Mambriani, A. Una interpretazione geometrica della variazione doppia totale. Atti Sem. Mat. Fis. Univ. Modena 1, 131–134 (1947).

Let $f(x, y)$ be a continuous function, with continuous first partial derivatives, in the rectangle $a \leqq x \leqq a'$, $b \leqq y \leqq b'$. Let S_x, S_y be the surfaces defined over R by the respective equations $z = \partial f/\partial x$, $z = \partial f/\partial y$. Then the area of the projection of S_x upon the (x, z)-plane is equal to the area of the projection of S_y upon the (y, z)-plane, and both of these areas are equal to the total variation, in the Vitali sense, of $f(x, y)$ over R.

T. Radó (Columbus, Ohio).

Theory of Functions of Complex Variables

Davydov, N. A. The continuity of an integral of Cauchy type in a closed region. Doklady Akad. Nauk SSSR (N.S.) 64, 759–762 (1949). (Russian)

The author proves the following. Let L be a closed rectifiable Jordan curve and $f(\zeta)$ a continuous function given on L; if the principal value integral

$$\Phi(\zeta) = \lim_{\delta \to 0} (2\pi)^{-1} \int_{L_\delta} \left| \frac{f(\zeta) - f(\zeta_1)}{\zeta - \zeta_1} \right| |d\zeta|$$

exists uniformly with respect to ζ_1 on L, then

$$(1) \quad (2\pi i)^{-1} \int_L \frac{f(\zeta)}{\zeta - z} d\zeta \to f(\zeta_0) + (2\pi i)^{-1} \int_L \frac{f(\zeta) - f(\zeta_0)}{\zeta - \zeta_0} d\zeta,$$

as z tends along any path to any point ζ_0 on L. The method of proof is different from that used in establishing a similar theorem due to Privaloff [same Doklady (N.S.) 23, 859–862 (1939); these Rev. 1, 305]. Corollary: the Cauchy integral

$$F(z) = (2\pi i)^{-1} \int_L \frac{f(\zeta)}{\zeta - z} d\zeta$$

is a continuous function in the closed region $\bar{G}$ bounded by the closed rectifiable Jordan curve L, provided

$$|\zeta_1 - \zeta_2| \geqq c |s(\zeta_1) - s(\zeta_2)|^{1+\beta},$$

$\beta \geqq 0$, and

$$|f(\zeta_1) - f(\zeta_2)| \leqq K |\zeta_1 - \zeta_2|^\alpha,$$

$\beta/(1+\beta) < \alpha \leqq 1$; the limiting values of F are given by (1). Also, $F(z)$ is continuous in $\bar{G}$ if $|f(\zeta_1) - f(\zeta_2)| \leqq K |\zeta_1 - \zeta_2|$ ((1) again applies). The latter result enables extension of the theory of singular integral equations (developed in the complex plane for piece-wise continuous curves) to the case of any rectifiable contour, provided $f \varepsilon$ Lip 1.

W. J. Trjitzinsky (Urbana, Ill.).

Carafa, M. Sul prolungamento analitico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 127–133 (1948).

The author first obtains a form of an expansion due to Mittag-Leffler [Atti del IV Cong. Internaz. Mat., Rome, 1908, v. 1, pp. 67–85 (1909)]. For any function f, regular at a, let $Q_n(f)$ be the polynomial whose value at z is

$$\sum_{k_1=0}^{n^1} \sum_{k_2=0}^{n^2} \cdots \sum_{k_n=0}^{n^{k_n}} \frac{f^{(k_1+k_2+\cdots+k_n)}(a)[(z-a)/n]^{k_1+\cdots+k_n}}{k_1! k_2! \cdots k_n!}.$$

Then, for any z in the star of f at a, $\lim Q_n(f)(z) = f(z)$. A function $P(t, z, \lambda)$ is defined such that $\lim_{\lambda \to 1} \partial P/\partial \lambda = 1/(t-z)$. It then follows that for any z in the star of f at a, inside Γ,

$$f(z) = \lim_{\lambda \to 1} \frac{\partial}{\partial \lambda} (2\pi i)^{-1} \int_\Gamma P(t, z, \lambda) f(t) dt.$$

R. C. Buck (Providence, R. I.).

Goodman, A. W. Note on regions omitted by univalent functions. Bull. Amer. Math. Soc. 55, 363–369 (1949).

The function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is regular and schlicht in $|z| < 1$. It is proved that, if $|c| \geqq \frac{1}{4}$ and $f(z)$ omits all values w with $|w - c| \leqq R$, then

$$R \leqq |c|(4|c| - 1)/(4|c| + 1)$$

(a best possible estimate). This result, however, is not new [see Rogosinski, Proc. Cambridge Philos. Soc. 35, 1–26

(1939)]. An estimate for the area of the set of all values w with $|w|<1$ taken by f in $|z|<1$ is also given.

W. W. Rogosinski (Newcastle-upon-Tyne).

Goluzin, G. M. On distortion theorems and the coefficients of univalent functions. Mat. Sbornik N.S. **23(65)**, 353–360 (1948). (Russian)

Let S be the class of functions $f(z)=z+c_2z^2+\cdots$ which are regular and univalent in $|z|<1$, and let Σ be the class of functions $F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$ which are univalent in $|\zeta|>1$ and regular except for a pole at $\zeta=\infty$. For these two classes of functions the author gives estimates of the type of the distortion theorem, and investigates the order of magnitude of all the coefficients c_n of f when the coefficients c_n for which n lies in an arithmetic progression are bounded above by a certain power of n.

Let F belong to Σ and let $\alpha_{\nu\nu'}$ $(\nu,\nu'=1,2,\cdots,n;\ n\geqq 1)$ be real quantities such that for real x_ν, $x_{\nu'}$ not all zero the quadratic form $\sum_{\nu,\nu'=1}^{n}\alpha_{\nu\nu'}x_\nu x_{\nu'}$ is positive. Then for any ζ_ν, $\zeta_{\nu'}$ $(\nu,\nu'=1,2,\cdots,n)$ lying in $|\zeta|>1$ we have

$$\prod_{\nu,\nu'=1}^{n}|1-(\zeta_\nu\bar{\zeta}_{\nu'})^{-1}|^{\alpha_{\nu\nu'}}\leqq\prod_{\nu,\nu'=1}^{n}|(F(\zeta_\nu)-F(\zeta_{\nu'}))/(\zeta_\nu-\zeta_{\nu'})|\leqq\prod_{\nu,\nu'=1}^{n}|1-(\zeta_\nu\bar{\zeta}_{\nu'})^{-1}|^{-\alpha_{\nu\nu'}}.$$

For $\nu=\nu'$ the difference quotient is to be interpreted as the derivative $F'(\zeta_\nu)$. The proof is based on Löwner's differential equation. Second, for F of class Σ and for any ζ_ν, $\zeta_{\nu'}$ $(\nu,\nu'=1,2,\cdots,n;\ n\geqq 2)$, lying on the circumference $|\zeta|=\rho$, $\rho>1$,

$$\prod_{\nu\neq\nu'}|(F(\zeta_\nu)-F(\zeta_{\nu'}))/(\zeta_\nu-\zeta_{\nu'})|\geqq(1-\rho^{-2})^{2(n-1)}\prod_{\nu\neq\nu'}|1-(\zeta_\nu\bar{\zeta}_{\nu'})^{-1}|^{1-2/n}.$$

Let f belong to S and write $f_{\mu,\nu}(z)=\sum_{n=0}^{\infty}c_{\mu n+\nu}z^{\mu n+\nu}$ $(\mu=1,2,\cdots;\ \nu=1,2,\cdots,\mu)$. Then

$$f(z)=f_{\mu,1}(z)+\cdots+f_{\mu,\mu}(z).$$

Suppose that for any μ, ν $(\mu\geqq 1;\ 1\leqq\nu\leqq\mu)$, the coefficients of the function $f_{\mu,\nu}(z)$ are $O(n^\alpha)$, $\alpha\geqq 0$. Then all the coefficients of $f(z)$ are $O(n^\alpha)$. *D. C. Spencer.*

Bazilevič, I. E. On distortion theorems and the coefficients of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **65**, 253–255 (1949). (Russian)

Let Σ be the class of functions $F(\zeta)=\zeta+\alpha_0+\alpha_1/\zeta+\cdots$ which are univalent in $|\zeta|>1$ and regular except for a pole at $\zeta=0$. Let S be the class of functions $f(z)=z+c_2z^2+\cdots$ which are regular and univalent in $|z|<1$. The author states certain results concerning these two classes of functions. If F belongs to Σ, then for any two points ζ_1, ζ_2 in $|\zeta|>1$,

$$|\log\{(F(\zeta_1)-F(\zeta_2))/(\zeta_1-\zeta_2)\}|\leqq\log\{|\zeta_1\zeta_2|(|\zeta_1|^2-1)^{-1}(|\zeta_2|^2-1)^{-1}\}.$$

In particular, for $|\zeta_1|=|\zeta_2|=\rho>1$,

$$(\rho^2-1)/\rho^2\leqq|F(\zeta_1)-F(\zeta_2)|/|\zeta_1-\zeta_2|\leqq\rho^2/(\rho^2-1),$$

where the lower bound was given by Golusin [Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 183–202 (1946); these Rev. **8**, 325] and the upper by Frank and Löwner [Math. Z. **3**, 78–86 (1919)]. Second, if $w=f$ belongs to S, then for any r, $0\leqq r<1$ and $x\geqq r\exp(\pi/e)$, the intersection of the circumference $|w|=x$ with the domain $D(r)$ which is the image of $|z|<r$ by f has a linear measure $l(r,x)$ which is not larger than the linear measure of the intersection of the same circumference with the domain $D^*(r)$ which is the image of $|z|<r$ by the function $f^*=z/(1-z)^2$. From this the author deduces the following asymptotic inequality. If $\alpha_n=\sup|c_n|$ in S, then $\limsup_{n\to\infty}(\alpha_n/n)\leqq\frac{1}{2}e$. *D. C. Spencer.*

Fridman, G. A. On the problem of the coefficients of functions of the class H_δ. Doklady Akad. Nauk SSSR (N.S.) **65**, 805–808 (1949). (Russian)

A function $f(z)$, holomorphic in $|z|<1$, is said to belong to the class H_δ if $\int_0^{2\pi}|f(\rho e^{i\varphi})|^\delta d\varphi<C(\delta,f)$ for all $\rho<1$. (I) If $f(z)\in H_\delta$, $0<\delta\leqq 1$, then

$$|a_n|<\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\varphi})|^\delta d\varphi\right\}^{1/\delta}(n+1)^{(1/\delta)-1},\qquad n=0,1,2,\cdots,$$

and $|a_n|=o((n+1)^{(1/\delta)-1})$. (II) If there exists a positive integer k such that $k\delta\geqq 2$ and the series

$$S\equiv\sum_{n=1}^{\infty}\left|\sum_{p=1}^{n}\frac{(-1)^p}{n!p!}\frac{d^n}{dz^n}\left[\left\{\frac{f(z)-f(0)}{f(0)}\right\}^p\right]_{z=0}\frac{\Gamma(p-1/k)}{\Gamma(-1/k)}\right|^{k\delta/(k\delta-1)}$$

is convergent, then $f(z)\in H_\delta$ and

$$\int_0^{2\pi}|f(e^{i\varphi})|^\delta d\varphi\leqq 2\pi|f(0)|^\delta(1+S)^{k\delta-1}.$$

To prove (I) let $f(z)=f_0(z)b(z)$, where $b(z)$ is a Blaschke product and $f_0(z)$ has no roots. The further decomposition $f_0(z)=[f_0(z)]^{\delta/\delta_1}[f_0(z)]^{\delta/2}$, $\delta/\delta_1+\delta/2=1$, permits the author to use Parseval's theorem on the second factor and induction from δ_1 to δ on the first factor. In carrying out the details of the proof, the Hausdorff-Young theorem [Hausdorff, Math. Z. **16**, 163–169 (1923)] plays an important role.

A. W. Goodman (Lexington, Ky.).

Mathéev, A. Sur les fonctions holomorphes dans le cercle-unité, dont les zéros ont leurs points limites sur la frontière. C. R. Acad. Bulgare Sci. Math. Nat. **1**, no. 2–3, 13–14 (1948).

Cf. the same C. R. **1**, no. 1, 29–32 (1948); these Rev. **10**, 363.

Leont'ev, A. F. On entire functions of exponential type assuming given values at given points. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 33–44 (1949). (Russian)

The author seeks conditions on a sequence $\{\lambda_n\}$ of complex numbers under which, for every sequence $\{a_{\pm n}\}$ satisfying $|a_{\pm n}|<e^{|\lambda_n|}$, there will exist at least one entire function $\omega(z)$ of exponential type such that (*) $\omega(\pm\lambda_n)=a_{\pm n}$. His principal results are as follows. Assume the obviously necessary conditions that $\{\lambda_n\}$ has no finite limit point and $\limsup n/\lambda_n<\infty$, and introduce $F(z)=\prod(1-z^2/\lambda_n^2)$. (1) The additional condition

$$\limsup|\lambda_n|^{-1}\log|1/F'(\lambda_n)|=\delta<\infty$$

is necessary and sufficient. (2) Let $F(z)$ be of type h and let $\limsup|\lambda_n|^{-1}\log|a_{\pm n}/F'(\lambda_n)|=\gamma$. Then for any positive ϵ there is at least one entire function $\omega(z)$ satisfying (*), of type less than $h+\gamma^++\epsilon$, where $\gamma^+=\gamma$ if $\gamma\geqq 0$, $\gamma^+=0$ if $\gamma<0$. This result is sharp in the sense that sequences $\{\lambda_n\}$ and $\{a_{\pm n}\}$ can be found for which there is no $\omega(z)$ of type less than $h+\gamma^+$, satisfying (*). If the λ_n are real and positive and $\lim n/\lambda_n=\sigma<\infty$, then $h+\gamma^++\epsilon$ can be replaced by $(\pi^2\sigma^2+(\gamma^+)^2+\epsilon)^{\frac{1}{2}}$. *R. P. Boas, Jr.* (Providence, R. I.).

Cohn, Harvey. Some Diophantine aspects of modular functions. I. Essential singularities. Amer. J. Math. 71, 403–416 (1949).

The author considers the behavior of certain modular forms of negative dimension on approach to the boundary of their domain of existence. He defines a "Poincaré theta function" of integral order $m \geqq 0$ and degree r to be a single-valued function $f(z)$, regular in the upper half-plane except for a finite number of poles, and satisfying

$$f(z') = (cz+d)^{2m} f(z), \tag{1}$$

where $z' = (az+b)/(cz+d)$ is a modular substitution; the integer r is the order of the pole in the Fourier series $f(z) = (\exp -2\pi i r z)(k_0' + k_1' \exp 2\pi i z + \cdots)$, $k_0' \neq 0$. In other words, f is a modular form of negative even dimension $-2m$, multiplier $v \equiv 1$, and having a pole at infinity of order r. Most of his results concern the Eisenstein series $F_{2m}(z) = \sum (cz+d)^{-2m}$, $m \geqq 2$, where (c, d) run over all integer pairs except $(0, 0)$, and the Poincaré series

$$G_{2m}(z) = \sum [cz^2 + (a+d)z + b]^{-2m}, \qquad m \geqq 2,$$

where the sum is extended over all substitutions $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the full modular group. As noted by Poincaré, $G_{2m}(z)$ vanishes identically unless $m = 6$, or $m \geqq 8$.

From (1) it is easily seen that $f(z) \asymp (z - p/q)^{-2m}$ as z approaches the rational point p/q in a Stolz neighborhood, i.e., inside the wedge $z = p/q + \rho_1 + i\rho_2$, ρ_1, ρ_2 real, $\rho_2 > 0$, $|\rho_2/\rho_1| \geqq \lambda > 0$. ($f \asymp g$ means that $|f/g|$ is bounded away from 0 and ∞.)

Using a generalization of a method of Hardy and Littlewood [Acta Math. 37, 193–239 (1914)], the author considers approach to an irrational point θ whose continued fraction has bounded partial quotients (this includes the case of real quadratic irrationalities). He proves that if $f(z)$ is a Poincaré theta function of order m without poles, then $f(z) = O[(z-\theta)^{-m}]$ as $z \to \theta$ in a Stolz neighborhood; if $f(z)$ has no zeros as well as no poles, $f(z) \asymp (z-\theta)^{-m}$. As a special case, $G_{2m}(z) = O[(z-\theta)^{-2m}]$, but $G_{12}(z) \asymp (z-\theta)^{-12}$. The last equation is especially interesting because G_{12} is proportional to the square of $\Delta = \eta^{24}(z)$, the Dedekind modular function. Thus $|\eta(z)|$ tends to infinity on approach to irrational θ in the class considered, whereas it is known to tend to zero on approach to all rational real points. The O-estimation can be improved to $\asymp$ only in the case noted above, i.e., in the case G_{12}.

Finally the author states without proof that a Poincaré theta function can satisfy $f(z) \asymp (z-\theta)^s$, s real, only if θ is rational and f is of degree zero, or if θ is irrational and $f(z) = G_{12}(z)^{k/12} \exp g(J(z))$. Here k is an integer, g is an entire function, and $J(z)$ is the absolute modular invariant.

J. Lehner (Philadelphia, Pa.).

Fourier Series and Generalizations, Integral Transforms

***Ríos, Sixto. Introducción a la Teoria de Series Trigonométricas. [Introduction to the Theory of Trigonometric Series].** C. Bermejo, Madrid, 1949. iv+193 pp.

Lectures given by the author in 1946–47, edited in collaboration with J. Béjar, T. Iglesias and M. E. Ríos.

R. P. Boas, Jr. (Providence, R. I.).

Malliavin, Paul. Sur la convergence absolue des séries trigonométriques. C. R. Acad. Sci. Paris 228, 1467–1469 (1949).

Let $r_1, r_2, \cdots$ be a sequence of positive numbers satisfying for each n the condition $r_n > r_{n+1} + r_{n+2} + \cdots$. Let P be the perfect and nondense set of all the points $x = \epsilon_1 r_1 + \epsilon_2 r_2 + \cdots$, where the ϵ_n take arbitrarily the values 0 and 1. The author shows that if a sine series $\sum \rho_n |\sin nx|$ converges in P, it converges uniformly there. *A. Zygmund* (Chicago, Ill.).

Zygmund, A. An example in Fourier series. Studia Math. 10, 113–119 (1948).

The Fourier series which represents $\prod_{k=1}^{\infty}(1 + ia_k \cos n_k x)$, with $n_{k+1}/n_k \geqq 3$ and a_k real, converges almost everywhere if $\sum_{k=1}^{\infty} a_k^2 < \infty$. However, if also $\sum_{k=1}^{\infty} |a_k| = \infty$ there is divergence on a set which is of the second category in every interval (and is nondenumerable). If there exist constants A and B such that $0 < A \leqq k|a_k| \leqq B$, the partial sums of the Fourier series are uniformly bounded. Earlier examples of Fourier series with uniformly bounded partial sums had only denumerable points of divergence. *P. Civin.*

Loo, Ching-Tsün. Note on the properties of Fourier coefficients. Amer. J. Math. 71, 269–282 (1949).

The starting point of this paper is a result of Hardy [Messenger of Math. 58, 50–52 (1928)] which states that if we consider a periodic Lebesgue integrable function $f(x)$ having the Fourier series $\sum a_n \cos nx$ (with the usual simplifications), and set $A_n = n^{-1} \sum_{k=1}^{n} a_k$, then whenever $f(x) \in L^p$, $1 \leqq p < \infty$, $\sum A_n \cos nx$ is again a Fourier series of the class L^p. This theorem is connected with that of M. Riesz which states that the series conjugate to a Fourier series of the class L^p, $1 < p < \infty$, is also of the class L^p. Riesz's result was extended by Zygmund to the case $p = 1$, where one has to assume $|f| \log^+ |f| \in L$. The aim of the author is to pursue this analogy further and consider extensions, à la Zygmund, of the above theorem of Hardy to the cases $p = 1$ and $p = \infty$. Series of the type $\sum A_n^* \cos nx$ are also considered, where $A_n^* = \sum_{k=n}^{\infty} a_k/k$. A typical result is: if $|f| \log^+ (1/|x|) \in L$ then $\sum A_n^* \cos nx$ is a Fourier series.

K. Chandrasekharan (Bombay).

Duffin, R. J., and Schaeffer, A. C. Functions whose Fourier-Stieltjes coefficients approach zero. Duke Math. J. 16, 327–329 (1949).

Consider the function $f(z) = \int_{-\pi}^{\pi} e^{izt} d\rho(t)$, where $\rho(t)$ is a normalized function of bounded variation. A typical result is as follows. Let k_n, $n = 1, 2, \cdots$, be a bounded sequence of real numbers, and suppose $f(n + ik_n) \to 0$ as $n \to \infty$. If $\rho(t)$ is continuous at π or at $-\pi$, then $f(x) \to 0$ as $x \to \infty$. This is a generalization of a result contained in works of Rajchman, Verblunsky, and R. C. Young which pertains to the case $k_n = 0$ [cf. R. C. Young, Math. Z. 40, 292–311 (1935)]. The proof is based upon a complex variable technique used in a previous paper of the authors [Amer. J. Math. 67, 141–154 (1945); these Rev. 6, 148]. *R. Bellman.*

Hartman, S. Sur les bases statistiques. Studia Math. 10, 120–139 (1948).

If $f(x)$ is a uniformly almost periodic function it has a distribution function

$$\rho(a) = \lim_{T \to \infty} T^{-1} \operatorname{meas} \{x | f(x) < a, 0 < x < T\}.$$

If the distribution function is continuous the limit is uniform in a and the minimum T beyond which the ratio

remains within ϵ of $\rho(a)$ for all a is an "ϵ statistical base" of $f(x)$. An upper bound for this minimum is found, in terms of the modulus of continuity of f and the function $L(\eta)$ with the property that every interval of length L contains a number τ with $|f(x+\tau)-f(x)|<\eta$ for all x. The discussion is then generalized to the consideration of ϵ statistical bases for finite aggregates of functions and a corresponding inequality is proved. Finally the upper bound of this theorem is evaluated explicitly for an f which is the sum of any two periodic continuous functions with incommensurable periods. *J. L. Doob* (Ithaca, N. Y.).

Baiada, Emilio. Una relazione trigonometrica e la velocità di gruppo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 355–362 (1948).

Let $A(k)$ be an even function of the real variable k which is absolutely continuous in $(-\infty, \infty)$. If $A'(k)$ is the derivative of $A(k)$ (which is known to exist almost everywhere), let $A'(k)/k$ be absolutely integrable over $(1, \infty)$. Then it is shown that

$$(2\pi)^{-1}\int_{-\infty}^{\infty} x \sin \xi x\, dx \int_{-\infty}^{\infty} A(k)\cos kx\, dk = -\tfrac{1}{2}[A'(\xi+0)+A'(\xi-0)]$$

whenever the two limits on the right hand side exist. This formula is applied to the consideration of a group of waves defined by

$$f(x,t)=(2\pi)^{-1}\int_{-\infty}^{\infty} A(k)\cos k[x-\phi(k)-V(k)t]dk.$$

If the centre of the group is defined by

$$\bar{x}=\int_{-\infty}^{\infty} xf^2(x,t)dx \Big/ \int_{-\infty}^{\infty} f^2(x,t)dx,$$

then

$$\bar{x}=\int_{-\infty}^{\infty} x^2f^2(x,t)dx \Big/ \int_{-\infty}^{\infty} A^2(k)dk.$$

The velocity of the centre of the group is $V_G=d\bar{x}/dt$. It is proved that

$$V_G=\int_{-\infty}^{\infty} A^2(k)\frac{d[kV(k)]}{dk}dk \Big/ \int_{-\infty}^{\infty} A^2(k)dk.$$

In the particular case when $A^2(k)$ is a Dirac delta function, this reduces to the ordinary formula $V_G=d[kV(k)]/dk$ for the group velocity. *E. T. Copson* (Dundee).

Lebedev, N. N. Some singular integral equations connected with integral representations of mathematical physics. Doklady Akad. Nauk SSSR (N.S.) **65**, 621–624 (1949). (Russian)

For two integral representation formulae of type

$$(1)\qquad f(x)=\int_0^{\infty}\omega(\tau)\varphi(x,\tau)d\tau\int_a^{\infty} f(\xi)\varphi(\xi,\tau)d\xi,$$

the author finds singular integral equations

$$(2)\qquad \varphi(x,\tau)=\mu(\tau)\int_a^{\infty} K(x,s)\varphi(s,\tau)ds,$$

with continuous spectrum $(0, \infty)$, having the $\varphi(x, \tau)$ as eigenfunctions. The cases are

(i) $\omega(\tau)=2\pi^{-2}\tau\sinh\pi\tau$, $\varphi(x,\tau)=x^{-\frac{1}{2}}K_{i\tau}(x)$, $a=0$

[N. N. Lebedev, C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 655–658 (1946); these Rev. **9**, 29], for which

$$K(x,s)=e^{-(x+s)}/(x+s);$$

and

(ii) $\omega(\tau)=\tau\tanh\pi\tau$, $\varphi(x,\tau)=P_{-\frac{1}{2}+i\tau}(x)$, $a=1$

[Mehler, Math. Ann. **18**, 161–194 (1881); V. A. Fock, C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 253–256 (1943); these Rev. **5**, 181], for which $K(x,s)=1/(x+s)$. In both cases $\mu(\tau)=\pi^{-1}\cosh\pi\tau$. [Notation as in Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 1944; these Rev. **6**, 64.] The expansion (1) is valid for any f satisfying, in case (i), f of bounded variation over any interval (a, b), $0<a<b<\infty$, $x^{-\frac{1}{2}}f(x)\log x\in L(0,\frac{1}{2})$, $f(x)\in L(\frac{1}{2},\infty)$, and in case (ii), f of bounded variation over (a, b) if $1<a<b<\infty$, $f(x)x^{-\frac{1}{2}}\in L(1,\infty)$. Formal solutions of the nonhomogeneous equations corresponding to (2) may be obtained from (1); details, with sufficient conditions for validity, are given in case (i). *J. L. B. Cooper* (London).

Ibragimov, I. I. On the completeness of systems of analytic functions $\{F(a_kz)\}$. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 45–54 (1949). (Russian)

The author uses the following criterion: the system $\{f_n(z)\}$ of analytic functions is complete in $|z|\leqq R$ if every real-valued function $g(s)$ of L^2, for which $\int_{|z|=R} g(s)f_n(z)ds=0$ for all n, is almost everywhere zero on $|z|=R$. He proves two theorems which contain previous results as special cases. (1) If $F(z)$ is bounded and analytic in $|z|<1$ and none of its derivatives vanishes at 0, if $|\alpha_k|\leqq 1$ and if $\sum(1-|\alpha_k|)$ diverges, then $\{F(\alpha_kz)\}$ is complete in $|z|\leqq 1$. (2) Let $F(z)$ be entire and let $F^{(k)}(0)\neq 0$, $k=0, 1, 2, \cdots$. Let $M(r)$ be its maximum modulus, $n(t)$ the number of α_k in $|z|<t$, $N(r)=\int_0^r t^{-1}n(t)dt$. Then $\{F(\alpha_kz)\}$ is complete in $|z|\leqq R$ if $\lim_{r\to\infty} N(r)/\log M(rR/\theta)>1$ $(0<\theta<1)$.

R. P. Boas, Jr. (Providence, R. I.).

Mitchell, Josephine. A summability theorem for double orthogonal series whose coefficients satisfy certain conditions. Amer. J. Math. **71**, 257–268 (1949).

Let $\{\varphi_{mn}(x, y)\}$ be a complete orthonormal system of real functions of class L_2 defined on the rectangle $\alpha\leqq x\leqq\beta$, $\gamma\leqq y\leqq\delta$ and let (*) $\sum_{m,n=1}^{\infty} a_{mn}\varphi_{mn}(x,y)$ be the corresponding Fourier orthogonal development of the function $f(x,y)\in L_2$. Let $s_{mn}=\sum_{i=1,j=1}^{m,n}a_{ij}\varphi_{ij}$ and $\sigma_{mn}=(mn)^{-1}\sum_{i=1,j=1}^{m,n}s_{ij}$. The result of this paper is as follows: if both the series

$$\sum_{m,n=1}^{\infty}[\log(m+1)]^{1+\epsilon}a_{mn}^2,\qquad \sum_{m,n=1}^{\infty}[\log(n+1)]^{1+\epsilon}a_{mn}^2$$

converge for $\epsilon>0$, then in order that the series (*) be summable $(C, 1, 1)$ to the sum A almost everywhere it is necessary and sufficient that the sequence $s_{2^p,2^q}$ converge to A almost everywhere in Q as p and $q\to\infty$.

K. Chandrasekharan (Bombay).

Rappoport, S. I. On certain transformations of summation formulas. Mat. Sbornik N.S. **24**(66), 87–100 (1949). (Russian)

Für eine auf $[a, b]$ messbare Funktion $f(x)$ sei das metrische Mittel die (eindeutig erklärte) Zahl $M(f; a, b)=h$, für die die Mengen $[f(x)\geqq h]$ und $[f(x)\geqq h+\epsilon]$, $\epsilon>0$, Masse $\geqq(b-a)/2$ bzw. $<(b-a)/2$ haben. Ferner sei

$$[f(x)]_N=\min\{|f(x)|, N\}\operatorname{sgn} f(x).$$

Das Interpolationspolynom $P_n'(x)$, das man aus dem Bernsteinschen Polynom $\sum_{\nu=0}^{n} f(\nu/n)\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}$ einer auf $[0, 1]$ messbaren Funktion f erhält indem man $f(\nu/n)$ durch $M(f;\nu n^{-1}-3n^{-1},\nu n^{-1}+3n^{-1})$ ersetzt, konvergiert nach L.

Kantorovitch [Rec. Math. [Mat. Sbornik] (1) 41, 503–510 (1934)] gegen $f(x)$ an jeder Stelle der approximativer Stetigkeit von f. Verf. betrachtet die entsprechende Frage für Interpolationsformeln

$$(*) \qquad A_n'(x)=\sum_{\nu=1}^{n} f(x_\nu^{(n)})\varphi_\nu^{(n)}(x),$$

wobei $x_\nu^{(n)}$ in $[a, b]$ liegende Knotenpunkte und $\varphi_\nu^{(n)}(x)$ gegebene Funktionen sind ($\nu=1, 2, \cdots, n$; $n=1, 2, \cdots$). Konvergiert $A_n'(x)\to f(x)$ gleichmässig für jede stetige Funktion und sind die $\varphi_\nu^{(n)}\geqq 0$, so gibt es zwei Folgen $\delta_n\to 0$, $\mu_n\to+\infty$ so dass

$$\bar{A}_n'(x)=\sum_{\nu=1}^{n}\varphi_\nu^{(n)}(x)M([f]_{\mu_n}; x_\nu^{(n)}-\delta_n, x_\nu^{(n)}+\delta_n)$$

die obige Konvergenzeigenschaft von $P_n(x)$ hat. Sind die $\varphi_\nu^{(n)}$ verschiedenen Vorzeichens, so ist die Behauptung nicht mehr richtig. Der Beweis beruht auf dem Lemma nach dem

$$\sum_{|x_\nu^{(n)}-x|>\lambda}\varphi_\nu^{(n)}(x)<\lambda$$

gilt für beliebige $\gamma, \lambda>0$, alle x und alle $n\geqq N(\gamma, \lambda)$.

Für integrierbare Funktionen behandelt Verf. noch andere Varianten der Formel (*), indem sie dort $f(x_\nu^{(n)})$ durch das arithmetische Mittel von $f(x)$ auf dem Intervall von $x_\nu^{(n+1)}$ bis $x_{\nu+1}^{(n+1)}$ ersetzt oder statt A_n' den Ausdruck $(d/dx)A_n^{r}(x)$, $F(x)=\int_a^x f dt$ betrachtet. Für Bernsteinsche Polynome fallen diese beiden Varianten zusammen; und Ref. [Rec. Math. [Mat. Sbornik] N.S. 2(44), 543–556 (1937)] hat gezeigt, dass sie gegen $f(x)$ konvergieren an jeder Stelle x, wo $F'(x)=f(x)$ ist. Nach Verf. [C. R. (Doklady) Acad. Sci. URSS (N.S.) 56, 9–10 (1947); diese Rev. 9, 89] haben auch gewisse andere Interpolationsformeln dieses Konvergenzverhalten. Verf. zeigt nun durch Beispiele, dass im allgemeinen Fall aus der Konvergenz einer der Varianten die Konvergenz der anderen nicht folgt. *G. G. Lorentz.*

Special Functions

Wendel, J. G. Note on the gamma function. Amer. Math. Monthly 55, 563–564 (1948).

The classical relation $\lim_{x\to\infty} x^{-a}\Gamma(x+a)/\Gamma(x)=1$ is established for real a and x by the use of Hölder's inequality

$$\int_0^\infty f(t)g(t)dt\leqq\left\{\int_0^\infty f(t)^p dt\right\}^{1/p}\left\{\int_0^\infty g(t)^q dt\right\}^{1/q},$$

$$p^{-1}+q^{-1}=1,$$

by taking $0<a<1$, $f(t)=e^{-t/p}t^{a/p}$, $g(t)=e^{-t/q}t^{(a-1)/q}$. First $1/p=a$, later $1/p=1-a$. *S. C. van Veen* (Delft).

Hsü, Hsien-Yü. On Sonine's integral formula and its generalization. Bull. Amer. Math. Soc. 55, 370–378 (1949).

The integral

$$S_m=\int_0^\infty t^{1+2r-mr}\prod_{n=1}^{m}J_r(a_n t)dt$$

has been evaluated for $m=3$ by Sonine [cf. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 1944, § 13.46(3); these Rev. 6, 64]. The author gives a proof of Sonine's result by using Hankel's inversion formula. He also obtains a formula for $m=4$ which is, however, equivalent with the one given in Watson [loc. cit., p. 414]. Some aspects of S_3 have been investigated by Dougall [Proc. Edinburgh Math. Soc. 37, 33–47 (1919)]; the author investigates S_4 in a similar manner. *A. Erdélyi.*

Persen, Leif N. Development of the Whittaker functions in terms of "incomplete" Bessel functions. Norsk Mat. Tidsskr. 28, 109–117 (1946). (Norwegian)

The "incomplete" Bessel functions are defined by

$$\Delta_n J_p(x)=\sum_{r=n}^{\infty}\frac{(-1)^r x^{p+2r}}{2^{p+2r}r!\Gamma(p+r+1)}=J^p(x)-\sum_{r=0}^{n-1}\frac{(-1)^r x^{p+2r}}{2^{p+2r}r!\Gamma(p+r+1)}$$

($n=0, 1, \cdots$). By elementary transformations the author obtains a result which, after some corrections, is

$$M_{k,m}(\xi^2)=\frac{\Gamma(2m+1)}{\Gamma(\frac{1}{2}+m-k)}k^{-m}e^{-\xi^2/2\xi}\sum_{n=0}^{\infty}\left\{\frac{-1}{k}\right\}^n\left\{\frac{2n-1}{2}+m\right\}\times\Gamma\left\{\frac{2n+1}{2}+m-k\right\}\Delta_n J_{2m}(2\xi\sqrt{k}).$$

This result will hold good for all values of k, m, ξ^2. The distinction between the cases k, m, ξ^2 real or imaginary, made by the author, is unnecessary. *S. C. van Veen* (Delft).

Tricomi, Francesco. Sulle funzioni ipergeometriche confluenti. Ann. Mat. Pura Appl. (4) 26, 141–175 (1947).

The Pochhammer-Kummer confluent hypergeometric function

$$F(a, c; x)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{h=0}^{\infty}\frac{\Gamma(a+h)}{\Gamma(c+h)}\frac{x^h}{h!}$$

is a particular solution of the differential equation

$$(1) \qquad xy''+(c-x)y'-ay=0.$$

By means of the "skew" Laplace-transformation $\mathfrak{L}_z^{(\varphi)}$ of the function $p(z)=z^{a-1}(1+z)^{c-a-1}$ along a straight line from the origin with argument φ, a new confluent solution $G(a, c; x)$ of (1) is obtained:

$$G(a, c; x)=\frac{1}{\Gamma(a)}\frac{1}{e^{2a\pi i}-1}\int_\infty^{(0+)} e^{-xz}z^{a-1}(1+z)^{c-a-1}dz$$

(a not an integer) and for $\Re(a)>0$

$$G(a, c; x)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-xz}z^{a-1}(1+z)^{c-a-1}dz.$$

The confluent functions $F(a, c; x)$ and $G(a, c; x)$ are connected by the relations

$$F(a, c; x)=\frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}H(a, c; x)+\frac{\Gamma(c)}{\Gamma(a)}x^{-(c-a)}e^x H(c-a, c; -x),$$

$|\arg(\pm x)|<\pi$, where $H(a, c; x)=x^a G(a, c; x)$, and

$$G(a, c; x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}F(a, c; x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}F(a-c+1, 2-c; x).$$

A simple asymptotic expansion is obtained for $G(a, c; x)$ and $H(a, c; x)$:

$$H(a, c; x)\sim\frac{1}{\Gamma(a)}\sum_{m=0}^{\infty}\binom{c-a-1}{m}\Gamma(a+m)/x^m.$$

For x and c real and positive, $\sqrt{x}>0$ the author obtains an

expansion of $F(a, c; x)$ by means of a series of Bessel functions:

$$F(-a, c; x/a) = \Gamma(c)x^{(1-c)/2}e^{hx/a}\sum_{\nu=0}^{\infty} A_\nu a^{-\nu}x^{\nu/2}J_{\nu+c-1}(2\sqrt{x}),$$

where h is an arbitrary nonnegative constant, and A_ν is defined by the generating function

$$f(z) = e^{az}\{1+(h-1)z\}^a/(1+hz)^{a+c} = \sum_{\nu=0}^{\infty} A_\nu z^\nu.$$

The asymptotic character of the last expansion is investigated, and from this an asymptotic property of the zeros ξ_m of $F(-a, c; \xi)$ is obtained by the following theorem. If N is an arbitrary positive integer, the function $F(-a, c; \xi)$ (a and c real and positive) has for $a > a_0$ (a_0 sufficiently large) at least N real positive zeros $\xi_1, \cdots, \xi_N$ satisfying the limit condition $\lim_{a\to\infty}(a\xi_m)^{\frac{1}{2}} = \frac{1}{2}j_{c-1,m}$ $(m = 1, 2, \cdots, N)$, where $j_{c-1,m}$ is the mth real positive zero of $J_{c-1}(z)$.

S. C. van Veen (Delft).

Shanker, Hari. An integral equation for Whittaker's confluent hypergeometric function. Proc. Cambridge Philos. Soc. **45**, 482–483 (1949).

The function $\phi(z) = z^{-\frac{1}{2}}W_{k,m}(z)$ satisfies the integral equation

$$\phi(x) = \frac{\Gamma(\frac{1}{2}-k+\nu)}{\Gamma(\nu-m)\Gamma(\nu+m)}\int_0^\infty (xy)^{\nu-\frac{1}{2}}(x+y)^{-\nu-\frac{1}{2}}W_{k,\nu}(x+y)\phi(y)dy$$

when $\Re(\nu \pm m) > 0$. Sharma's integral equation [J. London Math. Soc. **13**, 117–119 (1938)] is a particular case of this.

A. Erdélyi (Pasadena, Calif.).

Meixner, J. Asymptotische Entwicklung der Eigenwerte und Eigenfunktionen der Differentialgleichungen der Sphäroid-Funktionen und der Mathieuschen Funktionen. Z. Angew. Math. Mech. **28**, 304–310 (1948). (German. Russian summary)

Der Verfasser geht von der Differentialgleichung zweiter Ordnung aus, welche durch Separation der Wellengleichung in rotationselliptischen Koordinaten entsteht. Diese Differentialgleichung hat als Lösungen die Sphäroidfunktionen, welche als Spezialfälle die Kugelfunktionen, Zylinderfunktionen, Laguerreschen Funktionen und die Funktionen des parabolischen Zylinders umfassen. Verfasser gibt eine Entwicklung der Eigenfunktionen der Differentialgleichung nach Funktionen des parabolischen Zylinders. Mit Hilfe dieser Entwicklung gelingt es, für grosse positive oder negative Werte des Wellenparameters den Eigenwertparameter asymptotisch zu berechnen. Eine analoge Entwicklung wird vom Verfasser dadurch erhalten, dass er die Eigenfunktionen nach Laguerreschen Funktionen entwickelt. Mit Hilfe der von ihm aufgestellten asymptotischen Formeln berechnet Verfasser numerisch einige Eigenwertpaare und stellt dieselben durch Kurven dar. Zum Schluss zeigt Verfasser, dass diese Ergebnisse als Spezialfälle auch die entsprechenden Formeln für die Mathieusche Differentialgleichung enthalten. *M. J. O. Strutt* (Zürich).

Campbell, Robert. Sur la vibration d'un haut-parleur elliptique. C. R. Acad. Sci. Paris **228**, 970–972 (1949).

Verfasser betrachtet die Wellengleichung im Zusammenhang mit einem 3-achsigen Ellipsoid. Durch Separation der Variabeln entstehen drei Differentialgleichungen zweiter Ordnung, welche als Lamésche Wellengleichungen bezeichnet werden können. In der vorliegenden Arbeit befasst sich der Verfasser mit der Berechnung der Eigenwerte dieser Laméschen Wellengleichung nach einem Verfahren, das von Ince auf die Lamésche Potentialgleichung angewandt wurde [Proc. Roy. Soc. Edinburgh **60**, 47–63, 83–99 (1940); diese Rev. **2**, 46]. Er benützt eine analoge Transformation wie Ince und setzt als Lösung der entstandenen transformierten Differentialgleichung zweiter Ordnung mit einfach periodischen Koeffizienten, Fouriersche Reihen an. Für die Koeffizienten dieser Reihen entstehen rekurrente Beziehungen, welche für die Eigenwerte auf eine unendliche Determinante führen. Zur Berechnung des charakteristischen Exponenten der zuletzt genannten Differentialgleichung wendet Verfasser ein Verfahren von L. Brillouin an [Quart. Appl. Math. **6**, 167–178 (1948); diese Rev. **10**, 252]. Mit Hilfe dieses Verfahrens gelangt er zu einem Ausdruck für den charakteristischen Exponenten als Produkt zweier unendlicher, rasch konvergierender Reihen.

M. J. O. Strutt (Zürich).

Harmonic Functions, Potential Theory

Maruhn, Karl. Bemerkungen über das Verhalten von Potentialfunktionen im Unendlichen. Arch. Math. **1**, 253–262 (1949).

The author notes that parts of this paper are closely related to work by P. Painlevé [Leçons sur la résistance des fluides non visqueux, Gauthier-Villars, Paris, 1930] and to unpublished notes of P. Koebe. Let S be a closed surface within the sphere K with center at the origin of a Cartesian coordinate system in three-dimensional space. It is assumed that S has a continuously turning normal with direction cosines which satisfy a Lipschitz condition with exponent λ, where $0 < \lambda < 1$. Let $u(x, y, z)$ be a potential function which is regular in T_a (the set of all finite points exterior to S) and which is continuous, together with its normal derivative, on S. For the most general such u the author derives the following formula which, together with a corresponding representation in the two-dimensional case, forms the basis for much of the remainder of the paper:

$$u(x, y, z) = V(x, y, z) + M/R + C + \sum_{k=1}^{\infty}\sum_{\mu=1}^{2k+1} C_\mu^{(k)}H_\mu^{(k)}(x, y, z).$$

Here M is the "mass" of u, V is a potential function determined by u and S, and $H_\mu^{(k)}$, $\mu = 1, 2, \cdots, (2k+1)$, are linearly independent spherical harmonics of the kth order, $R = (x^2+y^2+z^2)^{\frac{1}{2}}$, and C is a constant. A number of theorems are given concerning entire potential functions and potential functions which are regular in T_a and satisfy certain conditions as $R = (x^2+y^2+z^2)^{\frac{1}{2}} \to \infty$. The hypotheses imposed by various authors in the study of problems of this type are compared. The theory is applied briefly to the treatment of the three boundary value problems of potential theory, from the point of view of integral equations.

F. W. Perkins (Hanover, N. H.).

Fichera, Gaetano. Applicazione della teoria del potenziale di superficie ad alcuni problemi di analisi funzionale lineare. Giorn. Mat. Battaglini (4) **2**(78), 71–80 (1948).

Comme dans un article antérieur [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 502–507 (1947); ces Rev. **10**, 192] l'auteur considère dans l'espace à 3 dimensions un domaine limité par des surfaces assez régulières: Σ_0 externe, $\Sigma_1, \cdots, \Sigma_p$ internes. Soit O_0 une origine quel-

conque, et $O_1, \cdots, O_p$ des points fixés dans les domaines bornés limités par les $\Sigma_1, \cdots, \Sigma_p$. On introduit pour chacun un système de fonctions harmoniques du type $Y_m\rho^m$ (pour O_0) ou Y_m/ρ^{m+1} pour O_i $(i>0)$; Y_m fonction de Laplace d'ordre m, ρ distance à O_i; à chaque m correspondent $2m+1$ fonctions de Laplace linéairement indépendantes. L'auteur reprend ces fonctions notées v_k et leurs dérivées normales sur l'ensemble-frontière F des Σ pour améliorer ses résultats antérieurs. Il montre que ces v_k forment un système fondamental au sens de Banach (relativement à la convergence uniforme des fonctions continues sur F), ce qu'il appelle d'ailleurs complet; mais j'ai déjà fait remarques dans l'analyse mentionnée plus haut, que ce résultat avait été donné pour un F plus général (avec critère exact de validité) par J. Deny [Bull. Soc. Math. France 73, 71–73 (1945); ces Rev. 7, 205], il est vrai, un peu sommairement et avec un lapsus aussitôt rectifiable dans l'énoncé du théorème qui doit être lu "Pour que toute fonction finie continue sur la frontière du compact E puisse être approchée. . . ." Mais Fichera montre de plus que les dv_k/dn forment un système fondamental dans le sous ensemble des fonctions continues sur F, de moyenne superficielle nulle. L'auteur se basant comme Deny sur l'équivalence des systèmes totaux et fondamentaux de Banach utilise des surfaces parallèles voisines de F et des propriétés, au voisinage de F, du potentiel de simple et double couche (distribution quelconque sur F), qu'il retrouve après G. C. Evans. Une application termine l'article. *M. Brelot* (Grenoble).

Westberg, R. On the harmonic and bi-harmonic problems of a region bounded by a circle and two parallel lines. Acta Polytech., no. 18, 66 pp. (1948).

[Also published as Ingeniörvetenskapsakademiens Handlingar, no. 197.] The paper consists of two parts. In part I the author determines the potential function outside a circular cylinder between two parallel planes when the values of the potential on the cylinder and the planes are arbitrary constants. In part II he studies the translation of the same cylinder in a viscous fluid between the two planes, in the direction normal to the planes. Whereas in part I the cylinder is not necessarily midway between the planes, in part II its displacement is assumed to be so small that it practically remains midway between the planes. The essential difference between parts I and II is that the first is based on the ordinary potential equation, and the second on the bi-harmonic differential equation.

To show the character of the author's work, let us consider the simplest problem of part I, that is, the cylinder (at unit potential) midway between the planes (both at potential zero). In units of the semi-distance of the planes, let λ be the radius of the cylinder. The problem is obviously two-dimensional (x, y rectangular coordinates, respectively normal and parallel to the cylinder axis). Then $V_0=\Re W_0$, where

$$W_0=\sum_{n=0}^{\infty}A_{2n}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\log\cot(\pi z/4),\quad z=x+iy,$$

satisfies the potential equation and the appropriate boundary conditions at infinity and at the two planes. The coefficients A_{2n} are to be determined from the condition that V_0 shall be equal to unity for $|z|=\lambda$. For that purpose $W_0+A_0\log(\pi z/4)$ is expanded in a Laurent series of z, which provides a Fourier expansion for V_0 in terms of the polar angle $\varphi=\arctan(y/x)$. Applying the boundary condition for $|z|=\lambda$ gives an infinite set of linear equations in the unknowns A_{2n}. The solution is obtained in the form of

$$A_0=\{\log(4/\pi\lambda)-\sum_{\nu=1}^{\infty}B_{4\nu}\lambda^{4\nu}\}^{-1},$$

$$A_{2n}=A_0\lambda^{4n}\sum_{\nu=0}^{\infty}A_{2n,4\nu}\lambda^{4\nu},\qquad n>0,$$

where $B_{4\nu}$ and $A_{2n,4\nu}$ are numerical constants, tabulated by the author for $n=1(1)6$, $\nu=0(1)7$ [4 or 5 S]; they satisfy certain recurrence formulas. The power series converge for $0\leqq\lambda<1$. In addition, the only singular points of $W_0(z)$ are shown to be confined to $-\beta_0\pm2m\leqq x\leqq\beta_0\pm2m$ $(m=0, 1, \cdots;$ $\beta_0=1-(1-\lambda^2)^{\frac{1}{2}})$. Various series expansions and tables are given to facilitate numerical calculations, and the question of convergence of all series is treated in detail.

Further topics considered are: electric capacity of the cylinder, electric force on the cylinder. In part II there is a table for the first eleven zeros of $z+\sinh z$, to nine decimals. There are references to many related papers by other authors. *C. J. Bouwkamp* (Eindhoven).

Differential Equations

Tanrikulu, Mahmut. Ordinary zero places in a body under plane stress. II. Rev. Fac. Sci. Univ. Istanbul (A) 13, 246–302 (1948).

Continuing a previous paper [same vol., 205–235 (1948); these Rev. 10, 340], the author studies in great detail and with many diagrams the solutions of the differential equation $y'/(1-y'^2)=P(x, y)/Q(x, y)$, where P and Q are homogeneous polynomials of degree n. *J. L. Synge* (Dublin).

Zimmerberg, H. J. The adjoint of Euler's linear differential operator. Amer. Math. Monthly 56, 332–334 (1949).

Haltiner, G. J. The theory of linear differential systems based upon a new definition of the adjoint. Duke Math. J. 15, 893–920 (1948).

The problem is the theory of the solutions of a linear ordinary differential equation (of any order), linear also in a disposable constant λ, when the solutions are defined by boundary conditions of general type given at two points $x=a, b$. As always the theory is linked with that of the corresponding adjoint differential equation and adjoint boundary conditions. The work relies on a definition of "adjoint" given by R. E. Langer [Trans. Amer. Math. Soc. 46, 151–190, 467 (1939); 53, 292–361 (1943); Amer. Math. Monthly 54, no. 7, part II (1947); these Rev. 1, 15; 4, 198; 9, 87].

The author is primarily concerned with the problem in its more formal and general aspect. He obtains the Green's function, develops an "orthogonal" property of the solutions and obtains the expansion of an arbitrary function in a series of "characteristic" solutions. A number of more precise problems arising from such expansions are indicated for further consideration. *T. W. Chaundy* (Oxford).

Železcov, N. A. The method of point transformation and the problem of the forced vibrations of an oscillator with "combined" friction. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 3–40 (1949). (Russian)

The basic system governing the oscillator is, in terms of a dimensionless time, (1) $\dot{x}=y$, $\dot{y}=-x+\varphi(x, y, g(t))+g(t)$,

where φ is friction and g the exterior force. The following assumptions are made: (A) g is piecewise continuous, bounded and has the period T. (B) For $\dot{x}=y\neq 0$, $\varphi=f(y)$, where f satisfies a Lipschitz condition in every finite interval. Moreover, setting $f(-0)=\alpha$, $f(+0)=\beta$, we have for $y=0$: $\varphi=\alpha,\beta$ for $x-g(t)\geqq\alpha$ and $\leqq\beta$, and $\varphi=x-g(t)$ for $\beta\leqq x-g(t)\leqq\alpha$. Finally for $y_1\neq y_2$ it is assumed that $(y_2-y_1)\{\varphi(x_2, y_2, g(t_2))-\varphi(x_1, y_1, g(t_1))\}<0$. Only continuous solutions of (1) are considered. The justification is that they are the only interesting solutions from the physical standpoint for an oscillator of positive mass. Setting $g_{max}-g_{min}=G$, the following results are proved.

(1) If $G\leqq\alpha-\beta$ there is a neutral segment σ:

$$g_{max}+\beta\leqq x\leqq g_{min}+\alpha,$$

and for $t\to+\infty$ every motion tends to a point of σ. (2) If $G>\alpha-\beta$, and if the oscillator has at least one bounded motion, then it has a periodic motion of period T, to which all motions converge as $t\to+\infty$. Hence in particular the periodic motion is stable. (3) If $G>\alpha-\beta$ and f is unbounded then there exists a bounded motion and theorem 2 is applicable. Given any motion $x(t)$, $y(t)$ let P_0 be the point $x(0)$, $y(0)$ and P_n the point $x(T_n)$, $y(T_n)$. The proofs of theorems 1, 2 rest upon examining $\lim_{n\to\infty}P_n$. The proof of theorem 3 consists in showing that there exists a Jordan curve Γ in the (x, y)-plane such that all the trajectories of (1) through its points cross it inwards and hence must be bounded as $t\to+\infty$.

The symmetrical oscillator is defined by: φ is an odd function of y and $g(t\pm T/2)=-g(t)$. Thus $f(-0)=f_0=-f(+0)$, $g_{max}=g_0=-g_{min}$. Thus the condition of theorem 1 is $g_0\leqq f_0$. (4) For a symmetrical oscillator if $g_0>f_0$, and (a) there exists a bounded motion, or else (b) $f(y)\to+\infty$ with y, there is a unique periodic motion $x(t)$, $y(t)$, as in theorem 2, and with the symmetry property $x(t\pm T/2)=-x(t)$, $y(t\pm T/2)=-y(t)$.

Extensive use is made by the author of Levinson's lemma [see reference below, p. 182] asserting that if $Q_1(t)$, $\{x_1(t), y_1(t)\}$ and $Q_2(t)$, $\{x_2(t), y_2(t)\}$ are any two solutions of the system then $d(Q_1(t), Q_2(t))$ is a monotonically decreasing function of t. [Relevant references: Eckolt, Z. Techn. Phys. 7, 226–232 (1926); Den Hartog, Philos. Mag. (7) 9, 801–817 (1930); Trans. A.S.M.E. 53, A.P.M. no. 9 (1931); Meissner, Z. Angew. Math. Mech. 15, 62–70 (1935); Ziegler, Ing.-Arch. 9, 50–76, 163–178 (1938); Levinson, J. Math. Physics 22, 41–48, 181–187 (1943); these Rev. 5, 66, 183; A. A. Andronow and C. E. Chaikin, Theory of Oscillations, Princeton University Press, 1949; these Rev. 10, 535; A. Liapounoff, Problème Général de la Stabilité du Mouvement, Princeton University Press, 1947; these Rev. 9, 34.] *S. Lefschetz* (Princeton, N. J.).

Zaremba, S. K. Nouvelle contribution à la discrimination des points singuliers des équations différentielles. Revista Ci., Lima 50, 185–192 (1948).

The differential equation $Y(x, y)dx-X(x, y)dy=0$ is studied in a neighborhood of a singular point of Kronecker order -1, 0, or $+1$. The more common assumptions of analyticity are lightened to requirement that partial derivatives of X and Y exist. The paper indicates results, geometric in character, of the type presented in more detail in earlier papers [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. 1934, 197–207; 1936, 439–445]. Neighborhoods of singular points of the three types indicated above are resolved into a finite number of sections of hyperbolic, parabolic, or elliptic character.

W. M. Whyburn (Chapel Hill, N. C.).

***Sommerfeld, Arnold. Partial Differential Equations in Physics. Translated by Ernst G. Straus.** Academic Press, Inc., New York, N. Y., 1949. xi+335 pp. $5.80.

Translation of Partielle Differentialgleichungen der Physik, Akademische Verlagsgesellschaft, Leipzig, 1947; these Rev. 10, 195.

Cinquini-Cibrario, M. Sopra i sistemi di equazioni alle derivate parziali a caratteristiche reali e multiple. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 4, 682–688 (1948).

The author has previously shown [Rend. Sem. Mat. Univ. Padova 17, 75–96 (1948); these Rev. 10, 539] the existence and uniqueness of the solution of the Cauchy problem for a hyperbolic system of first order differential equations

$$(1)\qquad F_i(x, y, z_1, \cdots, z_n, p_1, \cdots, p_n, q_1, \cdots, q_n)=0, \quad i=1, \cdots, n,$$

where $p_i=\partial z_i/\partial x$, $q_i=\partial z_i/\partial y$. Here the system was called "hyperbolic," if the characteristic equation

$$(2)\qquad \det|P_{ik}dy-Q_{ik}dx|=0,$$

with $P_{ik}=\partial F_i/\partial p_k$, $Q_{ik}=\partial F_i/\partial q_k$, has n distinct real roots dy/dx in the domain of variables x, y, z_i, p_j, q_k considered. She now extends the proof of existence and uniqueness to a system (1) which is hyperbolic in a more general sense, admitting multiple roots of (2). The assumption made is that the multiplicity of a root of (2) is independent of the variables x, y, z_i, p_j, q_k, and that for any root of multiplicity m the rank of the corresponding matrix $(P_{ik}dy-Q_{ik}dx)$ is exactly $n-m$. *F. John* (New York, N. Y.).

***Weinstein, Alexander. Separation theorems for the eigenvalues of partial differential equations.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 404–414. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

Consider the following two eigenvalue problems occurring in the Sturm-Liouville theory: (1) $v''+\lambda v=0$, $v(0)=v(\pi)$, $v'(0)=v'(\pi)$, and (2) $u''+\omega u=0$, $u'(0)=0$, $u'(\pi)=0$. If $\{\lambda_k\}$ and $\{\omega_k\}$ are the corresponding nondecreasing sequences of eigenvalues (multiplicity being taken into account), it is easily verified directly from these sequences that $\omega_k\leqq\lambda_k\leqq\omega_{k+1}$, $k=1, 2, \cdots$, i.e., the eigenvalues of (1) separate the eigenvalues of (2). The problems (1) and (2) may be restated in a form which lends itself to an immediate generalization to eigenvalue problems involving partial differential equations. Let f be a function defined only at 0 and at π by the equations $f(0)=-1$, $f(\pi)=1$, and let a be an unknown constant. Then the end conditions of (1) and (2) may, respectively, be rewritten:

$$v(0)f(0)+v(\pi)f(\pi)=0,$$
$$v_n(0)=af(0), \quad v_n(\pi)=af(\pi),$$

and

$$\partial u_n(0)=0, \quad \partial u_n(\pi)=0,$$

where the subscript n denotes differentiation with respect to the "outer normal" of the segment $(0, \pi)$. If S is a domain in the (x, y)-plane, C its boundary, and f a function defined on the boundary curve C and such that $\int_C f(s)ds=0$, then the corresponding partial differential eigenvalue prob-

lems are seen to be:

$$(DP_f) \qquad \Delta v+\lambda v=0, \quad \text{in } S;$$

$$\int_C fvds=0, \quad v_n(s)=af(s), \quad \text{on } C,$$

and the "base problem"

$$(DP_0) \qquad \Delta u+\omega u=0, \text{ in } S; \quad u_n=0, \text{ on } C,$$

where Δ is the Laplacian, n is the outer normal of C, and a is an unknown constant. The main purpose of the paper is to prove that the separation theorem $\omega_k \leqq \lambda_k \leqq \omega_{k+1}$, $k=1, 2, \cdots$, continues to hold, where $\{\lambda_k\}$ and $\{\omega_k\}$ are the eigenvalues of DP_f and DP_0, respectively. It is pointed out that the inequalities $\omega_k \leqq \lambda_k$ follow at once from the variational definition of the eigenvalues, but a proof is given which does not rely upon this. The proof is based on the remark that the solutions of the "intermediate problem" DP_f can be given explicitly in terms of the solutions of the base problem DP_0 [cf. A. Weinstein, Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques, Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937]. The argument proceeds by taking into account the eigenvalues which are common to DP_f and DP_0. As far as DP_f is concerned, two essentially different cases arise: $a=0$ and $a\neq 0$ (say $a=1$). Consider $a=0$ first, and let ω_k, $k>1$, be an eigenvalue of DP_0 of multiplicity r, with linearly independent eigenfunctions $u_1, \cdots, u_r$. Two mutually exclusive possibilities occur: either $\int_C fu_i ds=0$ for $i=1, 2, \cdots, r$ (ω_k is then said to be of class A), or there are at most $r-1$ linearly independent linear combinations $v=\sum_{i=1}^{r} c_i u_i$ such that $\int_C fvds=0$ (ω_k is then said to be of class B). Thus, if ω_k is of class A then it appears as an eigenvalue of DP_f of multiplicity at least r, and if ω_k is of class B then it appears as an eigenvalue of DP_f of multiplicity at least $r-1$ (as usual, "multiplicity zero" means "not an eigenvalue"). All additional eigenvalues of DP_f (i.e., with $a\neq 0$) are then characterized by showing that between any two consecutive eigenvalues of DP_0 of class B there is exactly one eigenvalue of DP_f with $a\neq 0$. These propositions, together with $0=\omega_1=\lambda_1<\omega_2$, imply the separation theorem.

J. B. Diaz (Providence, R. I.).

Eichler, Martin M. E. Analytic functions in three-dimensional Riemannian spaces. Duke Math. J. **16**, 339–349 (1949).

It has been shown recently [see, e.g., S. Bergman, same J. **14**, 349–366 (1947); M. Eichler, Abh. Math. Sem. Univ. Hamburg **15**, 179–210 (1947); these Rev. **9**, 181; **10**, 540] that all integrals of an elliptic partial differential equation of the second order in two variables, with analytic coefficients, can be constructed from analytic functions of a complex variable. The construction is accomplished by means of an integral operator which maps analytic functions of a complex variable into solutions of the partial differential equation. This approach is not restricted to two variables. E. T. Whittaker [Math. Ann. **57**, 333–355 (1903)] constructed all regular solutions of Laplace's equation in three dimensions by means of analytic functions, and this method has been developed by S. Bergman [Duke Math. J. **13**, 419–458 (1946); Trans. Amer. Math. Soc. **59**, 216–247 (1946); these Rev. **8**, 274; **7**, 448]. The present paper deals with elliptic linear partial differential equations of the second order in three variables, whose coefficients are analytic functions of the coordinates. Specifically, the following equation is considered:

$$(1) \qquad g^{\mu\nu}\nabla_\mu\nabla_\nu\phi+h^\mu\nabla_\mu\phi+k\phi=0,$$

in a Riemannian space with fundamental tensor $g^{\mu\nu}$; ∇_μ is the covariant differentiation operator, h^μ a contravariant vector, and k a scalar. The procedure is as follows. In the first place, "the Riemannian space is projected conformally on a surface." A complex-valued function $z=u+iv$ is said to be an analytic function in the Riemannian space R if it satisfies the equation $g^{\mu\nu}\nabla_\mu z\nabla_\nu z=0$. Any such function defines a "conformal projection" of R into a subset S of the complex number plane. For R of dimension 3, all conformal projections are characterized by means of a certain quasi-linear partial differential first order equation. In the second place, an operator is given which maps "analytic functions of a complex variable attached to the surface S" into solutions of (1). It is shown that all solutions of (1) can be obtained in this way. All results hold locally, i.e., in a sufficiently small neighborhood of any given point. In the last section it is shown that all conformal projections of Euclidean three-space can be described by a certain linear partial differential equation of first order in three variables.

J. B. Diaz (Providence, R. I.).

Olevskiĭ, M. N. Solution of the Dirichlet problem for the equation $\Delta u+\frac{p}{x_n}\frac{\partial u}{\partial x_n}=\rho$ for a hemispherical region. Doklady Akad. Nauk SSSR (N.S.) **64**, 767–770 (1949). (Russian)

Let D be the domain bounded by the hemisphere S_R: $\sum_{i=1}^{n} x_i^2=R^2$, $x_n>0$, and by the subset of the plane $x_n=0$ defined by the inequality $\sum_{i=1}^{n-1} x_i^2 \leqq R^2$, where $R>0$. This paper considers the Dirichlet problem in D for the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}+\frac{p}{x_n}\frac{\partial u}{\partial x_n}=\rho,$$

where p is a constant, and ρ is a continuous function defined on D. A fundamental solution of (1) (with $\rho=0$) is given explicitly, in terms of Gauss's hypergeometric function, for $n\geqq 2$, and $p\neq 2, 3, \cdots$. The simple remark that if u satisfies

$$\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}+\frac{p}{x_n}\frac{\partial u}{\partial x_n}=0,$$

then $v=x_n^p u$ satisfies the adjoint equation

$$\frac{\partial^2 v}{\partial x_1^2}+\cdots+\frac{\partial^2 v}{\partial x_n^2}-p\frac{\partial}{\partial x_n}\left(\frac{v}{x_n}\right)=0,$$

is used in determining Green's function for the given domain. A formal solution of (1), satisfying $u=\varphi$ on the boundary of D, where φ is a given continuous function on the boundary of D, is obtained in terms of Green's function, by employing Green's identity in the usual manner. It is shown that for $p<1$, and ρ continuously differentiable in D, the function u defined by the formula obtained is indeed a solution of the Dirichlet problem. For $n=3$, $p=-1$, this includes as a special case a result of C. Agostinelli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **26**, 216–221 (1937)]; and for $p=0$ the classical solution of the Dirichlet problem for Laplace's equation for a hemispherical domain. By letting $R\to\infty$, a formula for the Dirichlet problem for the half space $x_n>0$ is obtained, which contains as a special case a result of I. N. Vekua [Doklady Akad. Nauk SSSR (N.S.) **56**, 229–231 (1947); these Rev. **9**, 187] for $n=2$, $p<1$; and for $n\geqq 2$, $p=0$ the classical Poisson formula for Laplace's equation.

J. B. Diaz.

Bellman, R., Marshak, R. E., and Wing, G. M. Laplace transform solution of two-medium neutron ageing problem. Philos. Mag. (7) **40**, 297–308 (1949).

A point source of mono-energetic fast neutrons is situated at a height Z_0 above an infinite plane. This plane separates two media with different "slowing down" properties. Assuming that age theory [E. Fermi, Ricerca Sci. (2) 7_2, 13–52 (1936)] holds for both media, the equations for the slowing down densities are, in cylindrical coordinates, the rotationally symmetric heat conduction equations with different time scales in each media. These equations are to be solved subject to proportionality conditions on the function and its z-derivative at the joining plane. The authors solve this problem with the aid of the Laplace transform on the time variable and a spatial representation given by Sommerfeld in diffraction theory [J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, pp. 573–576]. Various approximations of the integrals involved in the solution are discussed. *A. E. Heins* (Pittsburgh, Pa.).

Amerio, Luigi. Relazioni tra il metodo della trasformata multipla di Laplace e il metodo di M. Riesz per l'integrazione di equazioni di tipo iperbolico. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 313–319 (1948).

The problem of Cauchy for the hyperbolic equation $E^k u = f$, where E denotes the operator

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_m^2} + \lambda,$$

is solved by means of the multiple Laplace transform, and the results are identified with those given by Riesz's method. The initial conditions are that u and its first $2k-1$ partial derivatives with respect to x_1 have given values (which may be taken to be zero without loss of generality) on a surface S, and the tangent hyperplane to S is nowhere characteristic. The retrograde part of the characteristic cone with vertex R and the surface S bound a domain ω. Then, by Green's theorem,

$$\int_\omega u E^k v d\omega = \int_\omega v f d\omega + \text{an integral over } \sigma;$$

here σ is the part of the boundary of ω which belongs to the retrograde cone, and the integral over σ involves the functions $u, Eu, E^2u, \cdots, E^{k-1}u$ and the canonical derivatives. If u is defined to be zero outside ω and if $v = \exp(-\sum_1^m p_h x_h)$, this equation becomes

$$(p_1^2 - \sum_2^m p_h^2 + \lambda)^k L(u) = L(f) + \text{an integral over } \sigma,$$

where $L(u)$ is the multiple Laplace transform over all space. Now if $k+1 > \frac{1}{2}m$, the function defined by

$$G(x_1, \cdots, x_m) = H_{k,m}(2r\lambda^{-\frac{1}{2}})^{k-\frac{1}{2}m} J_{k-\frac{1}{2}m}(r\lambda^{\frac{1}{2}}),$$

when $(x_1, \cdots, x_m)$ lies within the retrograde cone with vertex at 0 (r being the Lorentzian distance from 0), and $H_{k,m}$ is a constant depending only on k and m, and defined by $G(x_1, \cdots, x_m) = 0$ elsewhere, has the Laplace transform $(p_1^2 - \sum_2^m p_h^2 + \lambda)^{-k}$. Hence $L(u) = L(f * G) + \text{an integral over } \sigma$. From this it can be shown that $u = f * G$, a result equivalent to Riesz's formula

$$u(P) = H_{k,m} \int_{\omega_P} f(Q)(2r'\lambda^{-\frac{1}{2}})^{k-\frac{1}{2}m} J_{k-\frac{1}{2}m}(r'\lambda^{\frac{1}{2}}) d\omega,$$

where r' is the Lorentzian distance from Q to P. The case when $k+1 \leq \frac{1}{2}m$ can be reduced to the above by operating on the partial differential equation by E a sufficient number of times. *E. T. Copson* (Dundee).

Functional Analysis, Ergodic Theory

***Nachbin, Leopoldo. Topological vector spaces. I.** Notas de Matemática, no. 4. Livraria Boffoni, Rio de Janeiro, 1948. ii+102 pp. (Portuguese)

Après quatre paragraphes préliminaires consacrés à l'exposé des notions élémentaires indispensables d'algèbre et de topologie, l'auteur aborde la théorie des espaces vectoriels topologiques sous un aspect extrêmement général: le corps des scalaires peut en effet être un corps topologique commutatif quelconque, et ni l'espace vectoriel topologique ni le corps des scalaires ne sont supposés séparés; il donne en particulier la caractérisation des voisinages de 0 dans un espace vectoriel topologique, et un exposé très soigné de la théorie des parties bornées. Le paragraphe suivant est consacré à une étude très détaillée des corps commutatifs valués; l'auteur y démontre entre autres le critère de Shafarevitch-Kaplansky [Kaplansky, Duke Math. J. **14**, 527–541 (1947); ces Rev. **9**, 172] pour que la topologie d'un corps puisse être définie par une valeur absolue. Il introduit à ce propos la nouvelle notion de corps topologique strictement minimal; un tel corps K est caractérisé par la condition suivante: toute partie B de K telle qu'il existe un voisinage U de 0 pour lequel $1 \in UB$ est bornée dans K (c'est-à-dire telle que pour tout voisinage W de 0, il existe un voisinage V de 0 tel que $VB \subset W$). Un corps K dont la topologie peut être définie par une valeur absolue peut alors être caractérisé par les propriétés suivantes: (1) K est strictement minimal; (2) l'ensemble des x tels que $\lim_{n\to\infty} x^n = 0$ est un voisinage de 0. Si E est un espace vectoriel topologique sur un corps strictement minimal K, tout hyperplan fermé dans E peut être défini par une équation $f(x) = 0$, où f est une forme linéaire continue; il n'en est plus ainsi si K n'est pas strictement minimal. Les deux derniers paragraphes sont consacrés aux définitions et propriétés élémentaires des topologies faibles et des topologies fortes: l'auteur dit qu'une topologie sur un espace vectoriel E est forte si toute topologie plus fine pour laquelle les parties bornées sont les mêmes est identique à la topologie donnée [cf. Mackey, Trans. Amer. Math. Soc. **60**, 519–537 (1946); ces Rev. **8**, 519]. *J. Dieudonné.*

Köthe, Gottfried. Eine axiomatische Kennzeichnung der linearen Räume vom Typus ω. Math. Ann. **120**, 634–649 (1949).

Let K be a complete topological field and let D be a set of cardinal d. The vector spaces $O_d(K)$ of all functions from D to K and $F_d(K)$ of all functions from D to K with only a finite number of nonzero values are "dual" in an obvious sense. The author remarks that the weak topology in $O_d(K)$ defined by $F_d(K)$ makes $O_d(K)$ complete as a uniform structure with respect to the convergence of arbitrary Cauchy directed systems and shows that conversely every vector space over K complete in a weak topology is isomorphic to an $O_d(K)$. He shows furthermore that the cardinal d of D is not only the linear dimension of the conjugate space $F_d(K)$ but also is equal to a suitably defined continuous dimension of $O_d(K)$ itself. The closed subspaces of $O_d(K)$,

the continuous linear transformations of $O_d(K)$ into itself and other notions of dimension for $O_d(K)$ are also studied.

G. W. Mackey.

Mazur, S., et Orlicz, W. Sur les espaces métriques linéaires. I. Studia Math. **10**, 184–208 (1948).

The paper under review is the first of a series whose purpose is to generalize some Banach space theorems to the case of a locally convex topological vector space with a countable base of neighborhoods at the origin (a B_0-space in the authors' terminology). Their use of more general methods has also allowed the authors to extend the theory of linear equations to the theory of linear inequalities. In this paper, the authors have limited themselves to the presentation of some basic results and notions (for future reference) about topological vector spaces, and among other things they establish the following propositions. Let M be a module over the topological ring R such that: (a) R possesses a countable base of neighborhoods at zero, and (b) R is complete. Assume further that M is endowed with a topology such that: (a′) $x+y$ is a function continuous in both $x, y \in M$ and tx is a function continuous separately in $t \in R$ and in $x \in M$, and (b′) M possesses a countable base of neighborhoods at the origin. Then tx is continuous in both arguments (the proof consists of an application of the Baire category argument to R). If $\{T_k\}$ is a sequence of topologies on the (real or complex) vector space X, each of which is compatible with some (sub-additive and homogeneous) pseudo-norm, then $T=\cup_1^\infty T_k$ is compatible with some such pseudo-norm if and only if $T=\cup_1^K T_k$ for some integer $K \geqq 1$ ($\cup$ denotes the supremum in the complete lattice of all topologies on X, the ordering among topologies being given by the inclusion relation between classes of open sets). This proposition is used by the authors to prove that certain topological vector spaces cannot be normed; in an equivalent form, this was also observed by G. W. Mackey [see Trans. Amer. Math. Soc. **57**, 155–207 (1945), theorem VII-6; these Rev. **6**, 274]. A topological vector space possessing a countable base of neighborhoods at the origin has the property that the convex hull of every compact set is bounded if and only if it is locally convex. This is a converse to a well-known theorem due to Mazur [Studia Math. **2**, 7–9 (1930)]. Other properties which are equivalent to local convexity or to normability are also discussed. The paper includes a list of examples.

L. Nachbin.

Eidelheit, M. Quelques remarques sur les fonctionnelles linéaires. Studia Math. **10**, 140–147 (1948).

A linear variety M in a Banach space X is defined to be a translate of a closed linear manifold. The principal theorem states that the distance (x_0, M) between M and a point $x_0 \in X$ is the upper bound of the distances between x_0 and hyperplanes containing M. If X is a space conjugate to a space Y, the hyperplanes can be restricted to those of the form $x(y)=c$ for fixed $y \in Y$ provided that M is transfinitely closed. By using these theorems, the author shows that if $\{f_n\}$ is a sequence of linear functionals over X, a necessary and sufficient condition that a functional g be the strong limit of a sequence of linear combinations of the $\{f_n\}$ is that, for a bounded sequence $\{x_k\}$ in X, $\lim_{k\to\infty} f_n(x_k)=0$ for each n implies that $\lim_{k\to\infty} g(x_k)=0$. Applications are given to the moment problem and to conditions for uniqueness of solutions of infinite systems of linear equations.

R. E. Fullerton (Madison, Wis.).

Yood, Bertram. Transformations between Banach spaces in the uniform topology. Ann. of Math. (2) **50**, 486–503 (1949).

L'auteur étudie d'abord les relations entre une application linéaire continue T d'un espace de Banach X dans un espace de Banach Y, et les applications linéaires continues de X dans Y qui sont "voisines" de T au sens de la norme, lorsque T est une application de X sur Y ou un isomorphisme de X dans Y. Il donne ensuite une classification des applications linéaires continues de X dans Y basée sur des propriétés de nature algébrique; par exemple, il montre que, pour que T soit un isomorphisme de X dans Y, il faut et il suffit que, pour tout espace de Banach W, il n'existe aucune suite d'applications linéaires continues U_n de W dans X telles que $\|U_n\|=1$ et que TU_n tende vers 0 (au sens de la norme). Cette classification est modelée sur une classification analogue donnée par Rickart [Duke Math J. **14**, 1063–1077 (1947); ces Rev. **9**, 358] pour les éléments d'une algèbre de Banach; l'auteur compare la classification de Rickart à la sienne, lorsqu'on applique cette dernière aux applications linéaires $x \to ax$ ou $x \to xa$ d'une algèbre de Banach B (notées a^l et a^r respectivement). Il montre enfin que l'image de B par l'application $a \to (a^r)^r$ dans l'algèbre de Banach $B'=E(E(B))$ ($E(B)$ désignant l'algèbre des endomorphismes continus de B) a la propriété que pour toute algèbre de Banach B'' contenant B', la classification d'un élément $a \in B$ (identifié à $(a^r)^r$) est la même dans B' et dans B''.

La bibliographie est incomplète; par exemple, le corollaire 3.12 a déjà été démontré par le rapporteur [Bull. Sci. Math. (2) **67**, 72–84 (1943); ces Rev. **7**, 124]; de même le théorème 2.5 est un cas particulier d'un théorème du rapporteur [Ann. Sci. École Norm. Sup. (3) **59**, 107–139 (1942), théorème 11; ces Rev. **6**, 178].

Le rapporteur saisit cette occasion pour corriger une erreur bibliographique qui lui a été signalée dans deux de ses compte-rendus [ces Rev. **9**, 42, 241]: le théorème sur la compacité faible de la boule unité dans un espace de Banach, publié par N. Bourbaki [sans démonstration] [C. R. Acad. Sci. Paris **206**, 1701–1704 (1938)] se trouve déjà dans la thèse de L. Alaoglu soutenue à Chicago la même année, et est annoncé par ce dernier [Bull. Amer. Math. Soc. **44**, 196, 459 (1938)].

J. Dieudonné (Nancy).

Myers, S. B. Spaces of continuous functions. Bull. Amer. Math. Soc. **55**, 402–407 (1949).

Soit X un espace complètement régulier, $B(X)$ l'espace de Banach des fonctions numériques continues et bornées dans X, avec la norme $\|b\|=\sup_{x \in X} |b(x)|$. Continuant des recherches antérieures [Ann of Math. (2) **49**, 132–140 (1948); ces Rev. **9**, 291] l'auteur dit qu'un espace de Banach B opère de façon complètement régulière (en abrégé c.r.) sur X s'il existe un sous-espace G de $B(X)$ isométrique à B tel que pour tout ensemble fermé $K \subset X$ et tout point $x_0 \in K$, il existe $b \in G$ telle que $b(x_0)=\|b\|$ et $\sup_{x \in K} |b(x)| < \|b\|$. Utilisant ses résultats antérieurs [loc. cit.], il donne d'abord des conditions nécessaires et suffisantes pour qu'un espace de Banach opère c.r. sur un espace compact X. Il montre ensuite que si X peut être plongé dans un espace de Hilbert (séparable ou non) H, l'espace $B(X)$ contient un sous-espace G qui opère c.r. sur X et est image par une fonction linéaire continue de $R^2 \times H$. Enfin, il obtient les critères suivants: pour que X soit un espace métrique séparable (de dimension finie), il faut et il suffit qu'il existe un sous-

espace séparable (de dimension finie) de $B(X)$ qui opère c.r. sur X. *J. Dieudonné* (Nancy).

Milgram, A. N. Multiplicative semigroups of continuous functions. Duke Math. J. 16, 377–383 (1949).

Let X, Y be compact Hausdorff spaces and $C(X)$, $C(Y)$ the real continuous functions, taken as multiplicative semigroups. Then an isomorphism between $C(X)$ and $C(Y)$ induces a homeomorphism of X and Y; and it is described by that homeomorphism, together with multiplicative automorphisms of the real numbers which are continuous except at a finite number of isolated points. The fundamental idea in the proof is to recapture the points by a suitable kind of maximal ideal. *I. Kaplansky* (Chicago, Ill.).

Abdelhay, J. On a theorem of representation. Bull. Amer. Math. Soc. 55, 408–417 (1949).

It is first observed that the order and norm of the space BC of bounded functions on any set can be described by a single functional φ, $\varphi(x)=\max(0, \inf x_-)$, where x_- is the negative part of the function x. This functional is then axiomatized by listing certain of its properties, including an existence requirement of what turns out to be a "positive part" x_+ for each x in the linear space V in question. The composite object thus obtained is called an S-space, and the axioms are these: $-\varphi(x)$ is nonnegative, subadditive, positively homogeneous; $\varphi(x)+\varphi(-x)=0$ implies $x=0$; if x is given, an x_+ can be found such that $\varphi(x_+)=\varphi(x_+-x)=0$ with $\varphi(y)=\varphi(y-x)=0$ implying $\varphi(y-x_+)=0$; and a Cauchy condition: if $\varphi(x_m-x_n)\to 0$ there is an x such that $\varphi(x_n-x)$, $\varphi(x-x_n)\to 0$. Then V can be partially ordered by setting $x\geqq 0$ if $\varphi(x)=0$, and a variety of norms compatible with the ordering and making V complete can be introduced. Additional concepts are studied, especially that of S-ring; and there is presented a set of characteristic properties of those S-rings which can be represented by spaces $C_0(X)$, which are the continuous real functions on a compact Hausdorff space having a common zero. *R. Arens.*

Morse, Marston, and Transue, William. Integral representations of bilinear functionals. Proc. Nat. Acad. Sci. U. S. A. 35, 136–143 (1949).

The purpose of this note is to present a series of fundamental definitions and theorems which the authors propose to apply subsequently to the study of a certain bilinear sum which contains the classical second variation as a very special case. The theory of this bilinear sum is to be used to extend Hilbert space theory and the theory of integro-differential equations. The bilinear functionals considered in the present note are those defined on $A\times B$, where A and B coincide either with the space C of Banach or with the linear space L_2 of measurable functionals $x(s)$ defined for $0\leqq s\leqq 1$. For such functionals, there exists a representation in terms of a repeated L-S integral. Application of this representation to the study of the bilinear form mentioned above requires an extensive machinery which is developed in the present note. *T. Radó.*

Tseng, Yuan-Yung. Sur les solutions des équations opératrices fonctionnelles entre les espaces unitaires. Solutions extrémales. Solutions virtuelles. C. R. Acad. Sci. Paris 228, 640–641 (1949).

Let H be a possibly nonseparable Hilbert space over the real, complex or quaternion field. The operator A is distributive and closed. The paper is concerned with the existence of a solution or of a virtual solution of $Az=x$ and the characterization of the minimal solution. If y is the projection of x on the closed linear extension of the range, then a solution of $Az=y$ is called a virtual solution of $Az=x$. *D. G. Bourgin* (Urbana, Ill.).

Schäfke, Friedrich Wilhelm. Eine Eigenschaft der charakteristischen Kurven gewisser Eigenwertaufgaben mit zwei linearen Parametern. Math. Nachr. 1, 62–65 (1948).

The author considers the eigenvalue problem

$$Ay-\beta By-\gamma Cy=0,$$

where A, B, C are Hermitian operators defined on a subspace of a vector space with a Hermitian inner product. He assumes that to any real γ there corresponds an at most enumerable sequence of real eigenvalues $\beta_\nu(\gamma)$ and to any real β an at most enumerable sequence of real eigenvalues $\gamma_\nu(\beta)$, together with some further conditions. He shows that if the real numbers β_1^*, γ_1^* and β_2^*, γ_2^* are not pairs of eigenvalues, then

$$\prod_\nu \frac{\beta_1^*-\beta_\nu(\gamma_1^*)}{\beta_2^*-\beta_\nu(\gamma_2^*)}=\prod_\nu \frac{\gamma_\nu(\beta_1^*)-\gamma_1^*}{\gamma_\nu(\beta_2^*)-\gamma_2^*}.$$

Applications to the theory of Mathieu functions are promised for a later paper. *F. Smithies* (Cambridge, England).

Hille, Einar. Les semi-groupes linéaires. C. R. Acad. Sci. Paris 228, 35–37 (1949).

The author defines a semigroup $\{T(a)\}$ of linear operators on a complex Banach space to itself dependent on n parameters, $a=\{a_i \mid i=1, \cdots, n\}$, which are represented by the nonnegative coordinates of n-dimensional Euclidean space. He gives an ordered set of conditions such that the first two are sufficient for measurability of $T(a)$ to imply continuity in the parameter space away from frontier points. Other conditions guarantee the existence of infinitesimal generators and finally the analogues of the three fundamental theorems of Lie for continuous groups can be asserted. *D. G. Bourgin* (Urbana, Ill.).

Freundlich, Marianne. Completely continuous elements of a normed ring. Duke Math. J. 16, 273–283 (1949).

The author considers a commutative normed ring R with an identity element e. An element $u\in R$ is called completely continuous if the operator $T_u: T_u(x)=ux$ is a completely continuous operator on the Banach space R. A number of well-known results for operators on a Banach space are formulated and applied to the present case. For example, if $t=e-u$, where u is completely continuous and does not possess an inverse in R, then there exists an integer N such that R is the direct sum of the principal ideal t^NR and the ideal A of annihilators of t^N. An element w is called weakly completely continuous provided, for every bounded set $B\subset R$, wB is weakly conditionally sequentially compact. The set J of all weakly completely continuous elements of R is a closed ideal. Also $J=R$ if, and only if, R is reflexive. *C. E. Rickart* (New Haven, Conn.).

***Nemyckiĭ, V. V., and Stepanov, V. V. Kačestvennaya Teoriya Differencial'nyh Uravnenii. [Qualitative Theory of Differential Equations].** OGIZ, Moscow-Leningrad, 1947. 448 pp.

This is a self-contained exposition of basic topics in qualitative dynamics. Dynamics is concerned with differ-

ential equations of the form $dx/dt=f(x)$, where x and $f(x)$ are n-vectors. Such an equation defines a flow (one-parameter transformation group) in a natural way and this flow permits the consideration of topological and measure-theoretic properties of the differential equation. Letting A stand for analysis, T for topology and M for measure theory, we may briefly describe qualitative dynamics as concerned with the connections, under the action of the flow, between A, T, M on the one hand and T, M on the other.

The introduction [30 pages] contains a brief historical survey, the elementary existence theorems of differential equations, and a preliminary study of orbits in the plane and torus. Chapters 1, 2 and 3 are mainly concerned with connections between A and T. Chapters 1 and 2 deal with fixed (singular) points in the plane and n-space, respectively, while chapter 3 deals with periodic points. Work of Poincaré, Lyapunov, G. D. Birkhoff, etc., on classification of fixed and periodic points, analytic criteria for their types, various kinds of stability, etc., is presented here. Chapter 4 is concerned with connections between T and T, relative to a flow acting on a metric space. The chapter begins with an elementary discussion of metric spaces. It presents Birkhoff's theory [central motions, minimal sets, etc.; see his Dynamical Systems, Amer. Math. Soc. Colloquium Publ., v. 9, New York, 1927, chap. 7] and several other topics of a related nature. Chapter 5 is concerned with connections between T and M, relative to a flow in a metric measure space. The chapter opens with an elementary discussion of measure. The various recurrence theorems (Poincaré, Hopf, etc.), the ergodic theorem in its different forms, and theorems (Krylov and Bogolyubov) on the existence of an invariant measure are proved. A selective bibliography contains about 130 references; the Russian entries increase its value. There is an index. Nemyckiĭ wrote chapters 1, 2 and 3, and Stepanov wrote the introduction and chapters 4 and 5.

The style is quite clear and details of proof are copiously supplied. Many illuminating examples are fully analyzed. In spite of its length, the book is obviously intended as a textbook and is not exhaustive in any direction. With few and relatively minor exceptions, its contents are to be readily found in the American and Western European literature.

W. H. Gottschalk (Philadelphia, Pa.).

Theory of Probability

***David, F. N. Probability Theory for Statistical Methods.** Cambridge University Press, 1949. ix+230 pp. $3.50.

This is intended as an elementary textbook in the theory of probability for students of statistics. Two preliminary chapters are followed by four chapters on the binomial distribution. Successive chapters deal with probabilities a posteriori, confidence limits, genetical applications, Markov estimates, the multinomial distribution, random variables, characteristic functions, and moments of common statistics. Thus, in spite of its name, this is a book on probability and statistics. The distinction between the theory of probability and the statistical theory of estimation is not made, and the general effect is such as to heighten the confusion in the mind of the reader. For example, on page 71 the author gives the proof of a perfectly innocuous result, usually known as Bayes' theorem, about whose validity there is no argument. She precedes this proof by the warning that "the modern point of view is that, strictly speaking, the theorem is wholly fallacious except under very restricted conditions." The reference here is obviously to statistical applications of the theorem. Again on page 72 she states that "unless . . . the prior probability of the event E_t is known, the formula cannot be valid (sic)." The validity of a formula can hardly depend upon knowledge of the particular values of the quantities which enter into it. The author is obviously very enthusiastic and devoted to her task. The examples are many, frequently original, lively, and interesting. *J. Wolfowitz* (New York, N. Y.).

Göring, Emil. Definition und Bestimmung der Wahrscheinlichkeit durch des Kollektiv allgemeiner Art. Mitt. Verein. Schweiz. Versich.-Math. **48**, 145–170 (1948).

In certain actuarial problems it is impossible to obtain sufficient data to make a direct estimate of the probabilities. In order to have a theoretical basis of procedure in such situations the author generalizes the concept of collective in such a manner that probabilities are related to limits of success ratios but are not determined by his generalized collective. *A. H. Copeland* (Ann Arbor, Mich.).

Wyss, Hans. Erwägungen über abhängige und unabhängige Wahrscheinlichkeiten. Mitt. Verein. Schweiz. Versich.-Math. **48**, 171–205 (1948).

The author considers the probability that an individual who possesses a set of n properties at time 0 will possess a specified subset of these properties at time t. Since there are 2^n subsets of the given set there are 2^n related probabilities to be considered. Each of these probabilities is a function of t. The loss of one property may or may not influence the loss of others. These questions arise in connection with joint life insurance policies. *A. H. Copeland.*

Cheng, Tseng Tung. The normal approximation to the Poisson distribution and a proof of a conjecture of Ramanujan. Bull. Amer. Math. Soc. **55**, 396–401 (1949).

An estimate of the error term in the normal approximation to the Poisson distribution is given by adapting the method Uspensky applied to the binomial distribution. If $x=\lambda^{-\frac{1}{2}}(n-\lambda+\frac{1}{2})$ then

$$\sum_{r=0}^{n}\lambda^r e^{-\lambda}/r!=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x}e^{-t^2/2}dt+\tfrac{1}{6}(2\pi\lambda)^{-\frac{1}{2}}(1-x^2)e^{-x^2/2}+\delta,$$

where $|\delta|<.076\lambda^{-1}+.043\lambda^{-\frac{1}{2}}+.13\lambda^{-2}$. A conjecture of Ramanujan, proved by Szegö, is improved and proved.

M. Loève (Berkeley, Calif.).

Chung, K. L., and Hunt, G. A. On the zeros of $\sum_1^n \pm 1$. Ann. of Math. (2) **50**, 385–400 (1949).

A study of number of zeros in the sequence $\{\sum_1^n X_r\}$, where X_r are independent random variables which can take only two equiprobable values $+1$ and -1. Let W_r be the subscript of the rth zero. (I) If $\psi(y)\downarrow 0$ and $\int_1^\infty y^{-1}\psi(y)dy<\infty$ or $=\infty$, respectively, then $P\{W_r>2r^2/\psi^2(r)\text{ i.o.}\}=0$ or 1, respectively (i.o. stands for "infinitely often"). (II) If $\psi(y)\uparrow\infty$, $y/\psi(y)\uparrow\infty$ and $\int_1^\infty \psi(y)y^{-1}e^{-\psi^2(y)/2}dy<\infty$ or $=\infty$, respectively, then $P\{W_r<2r^2/\psi^2(r)\text{ i.o.}\}=0$ or 1, respectively. (III) Properties of changes of sign in the sequence considered are reduced to those of the number of zeros.

M. Loève (Berkeley, Calif.).

Good, I. J. The number of individuals in a cascade process. Proc. Cambridge Philos. Soc. 45, 360–363 (1949).

The author calculates the generating functions in a cascade (birth) process of the total number of descendants in r generations, of the number n_r of descendants in the rth generation, assuming immortals who continue to reproduce, and of $\sum_{r=1}^{s} n_r$. *J. L. Doob* (Ithaca, N. Y.).

Kac, M. On deviations between theoretical and empirical distributions. Proc. Nat. Acad. Sci. U.S.A. 35, 252–257 (1949).

Let N be a chance variable which has the Poisson distribution with mean μ. Let $X_1, X_2, \cdots$ be an infinite sequence of chance variables with the common continuous distribution function $\sigma(u)$. It is assumed that $N, X_1, X_2, \cdots$ are all independent. Define

$$\sigma_\mu^*(u) = \mu^{-1} \sum_{j=1}^{N} \psi_u(X_j),$$

where $\psi_u(x) = 0$ if $x > u$, $\psi_u(x) = 1$ if $x < u$ (if $N = 0$, $\sigma_\mu^*(u) \equiv 0$). The author proves that

$$\lim_{\mu \to \infty} P\{ \sup_{-\infty < u < \infty} |\sigma_\mu^*(u) - \sigma(u)| < \alpha/\sqrt{\mu} \} = 4\pi^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-(2k+1)^2 \pi^2/(8\alpha^2)}.$$

Let $\sigma_n(u) = n^{-1} \sum_{j=1}^{n} \psi_u(X_j)$. Let $y(u)$ $(0 \leqq u \leqq 1)$ be a Gaussian process such that $y(0) = y(1) = 0$, and let

$$E\{y(u)y(v)\} = \min(u, v) - uv.$$

Let $V_u(x) = 1 - \psi_u(x)$ and define

$$K_n = \int_{-\infty}^{\infty} V_u[n^{\frac{1}{2}}\{\sigma_n(u) - \sigma(u)\}] d\sigma(u).$$

The author proves that

$$\lim_{n \to \infty} P\{K_n < \alpha\} = P\left\{ \int_0^1 V_u\{y(u)\} du < \alpha \right\}.$$

[An obvious typographical error occurs in equation (11), which should begin with $\lim_{\mu \to \infty} \lim_{r \to \infty}$.]

J. Wolfowitz (New York, N. Y.).

TOPOLOGY

***Wilder, Raymond Louis. Topology of Manifolds.** American Mathematical Society Colloquium Publications, vol. 32. American Mathematical Society, New York, N. Y., 1949. ix+402 pp. $7.00.

The program to which this volume is devoted is based on the study, initiated in the main by Schoenflies, of positional invariants of sets in the plane or 2-sphere. A positional invariant of an A with respect to a set B is a property which is shared by all homeomorphic images of A which are contained in B. The best known example of such a positional invariant is embodied in the Jordan curve theorem: a simple closed curve in the 2-sphere has precisely two complementary domains and is the boundary of each of them. But more important, from the point of view of this book, is a converse, proved by Schoenflies, to the Jordan curve theorem: a subset of the 2-sphere is a simple closed curve if it has two complementary domains, is the boundary of each of them, and is accessible from each of these domains. Schoenflies used this same positional invariant, accessibility, to obtain other results, for example to characterize those subsets of the plane which are locally connected. The work thus initiated by Schoenflies was continued by Brouwer and Alexandroff, among others, and was brought substantially to completion as far as subsets of the plane and 2-sphere were concerned by the Polish and American schools of topologists in the third decade of this century. Their methods were for the most part set-theoretic and were not sufficient to handle similar problems in higher dimensions. Indeed, as Schoenflies had guessed and as the Alexander duality theorem indicated, homology theory was to prove necessary for these higher dimensional investigations. It was easy to translate the concepts of local connectedness and accessibility into the language of homology, using zero-dimensional cycles. Hence there was an immediate generalization of these concepts to higher dimensions by replacing zero by n. In addition, a large number of other definitions suggested themselves, and the relations between them led to a body of results which not only were useful for the extension of Schoenflies' work but also had many interesting connections with other topological questions.

The first extension of Schoenflies' converse to the Jordan curve theorem was given by Wilder in 1930. He showed that a subset of Euclidean 3-space whose complementary domains satisfied certain homology conditions was a 2-sphere. The method of proof was to show that the given properties of the complementary domains implied certain homology properties of the original set, and these in turn were sufficient to prove the original set a 2-sphere. Since, as is well known, there are no topological characterizations of any kind of the sphere or manifold of dimension greater than two, this indicated that there was little hope of a direct extension of this theorem to still higher dimensions. The way around this difficulty was to replace the classical manifold by a generalized manifold, a space having the same homology properties, in the large and in the small, as a classical manifold. This type of space had already been investigated by Čech, Lefschetz, and others from another point of view, and Wilder was able to show that an extensive theory of positional invariants could be worked out within the framework of these generalized manifolds, thus achieving a satisfactory generalization of Schoenflies' results to the n-dimensional case.

The general plan of this book follows the above brief historical outline. The first four chapters are devoted to such topics as basic concepts of topology, topological spaces, local connectedness, Peano spaces, and various characterizations of the arc, simple closed curve, 2-sphere, and 2-manifold. These chapters serve a double purpose: they contain basic material necessary for the work of the following chapters and they also furnish a motivation for the higher-dimensional positional invariants introduced later. Chapter V is devoted to combinatorial topology. Although it is not exhaustive, since it only considers those topics needed in the rest of the book, it contains a surprising amount of material. Chapters VI through IX take up the topological concepts necessary for the definition of a generalized manifold and then discuss the general properties of such spaces. The final three chapters are devoted to the study of positional invariants of subsets of generalized

manifolds. Much of the material here has appeared up to now only in the form of abstracts.

We now give a more detailed outline by listing the main topics discussed in each chapter. Chapter I. Basic topological concepts. Connectedness and related concepts. Characterization of the arc and simple closed curve. Chapter II. Locally connected spaces. Further characterizations of the arc and simple closed curve in terms of local connectedness. Unicoherence and related properties. Mod 2 combinatorial topology of the n-sphere, à la Alexander, and the Jordan-Brouwer-Alexander theorem. Uniform local connectedness and accessibility. The Schoenflies characterization of the simple closed curve in the 2-sphere. Chapter III. Characterization of a Peano space as a continuous image of the interval. Cyclic element theory. Applications to the characterization of the 2-sphere, closed 2-cell, and 2-manifold. Chapter IV. Connectedness and local connectedness in nonmetric spaces. Property S and its relation to local connectedness and uniform local connectedness. Accessibility. Applications to the study of the boundaries of domains in a 2-sphere. Chapter V. Homology theory for complexes. Čech homology theory. Applications to connectedness and local connectedness. Cohomology and products. Homologies in noncompact spaces and approximate homologies.

Chapter VI. Higher dimensional local connectedness and its influence on the homology structure of spaces. Higher dimensional local coconnectedness (i.e., local connectedness with respect to cocycles). Various relations between these properties. Chapter VII. Applications of homology theory to the study of continua. Separation of continua by local subsets. Higher dimensional analogues of non-cut-points and avoidable points, and their application to give further characterizations of the simple closed curve, 2-sphere, and 2-manifold. Further local homology invariants and relations between them. Property S_n (a generalization to higher dimensions of Sierpiński's property S) and its relations with local connectedness. Avoidable sets. Chapter VIII. Generalized manifolds. Orientability. Duality and linking theorems in generalized manifolds.

Chapter IX. Characterizations of generalized manifolds in terms of avoidability properties and in terms of other local positional invariants. Nonorientable generalized manifolds. Local orientability. Comparison of generalized manifolds with ordinary manifolds. Chapter X. Positional invariants of subsets of a generalized manifold. Uniform local connectedness and coconnectedness of open subsets, with an application to a generalization of the Jordan curve theorem and its converse. Further positional properties and the study of the boundary of a domain in a generalized manifold. Characterizations of $(n-1)$-dimensional generalized manifolds as domain boundaries in n-dimensional generalized manifolds. Chapter XI. Dualities between S properties of a closed subset of a generalized manifold and its complement. Dualities between S properties and local connectedness. Applications to characterizations of locally connected subsets in terms of S properties of complements and to the study of the boundaries of complementary domains of locally connected subsets. Dualities between S properties in terms of cocycles. Relations between avoidable properties and S properties and further characterizations of locally connected subsets. Weak S properties and weak uniform local connectedness. Applications to the characterization of locally connected domain boundaries and of sets whose complementary domains are bounded by generalized manifolds. Chapter XII. Various types of accessibility and their relations with other positional invariants. Applications to characterizations of submanifolds of a generalized manifold and of domains whose boundaries are generalized manifolds.

There are an appendix (listing a number of interesting unsolved problems), a good index, a very useful index of special symbols, and a bibliography. There are only a few pictures in the book, but there is a wealth of interesting and valuable examples which are described in sufficient detail to allow the reader to draw his own pictures with ease.

E. G. Begle (New Haven, Conn.).

Frucht, Robert. On the groups of repeated graphs. Bull. Amer. Math. Soc. **55**, 418–420 (1949).

Let a permutation group $\mathfrak{H}$, of order h and degree n, be applied to the elements of each row of a matrix of m rows and n columns, and let the rows be permuted according to the symmetric group of degree m. These $m!h^m$ permutations form a group $\mathfrak{S}_m[\mathfrak{H}]$ which was described by Pólya [Acta Math. **68**, 145–254 (1937), in particular, p. 178]. The author points out that if $\mathfrak{H}$ is the group of a graph having n nodes, $\mathfrak{S}_m[\mathfrak{H}]$ is the group of m separate copies of that graph, considered as a single graph. It is therefore also the group of the complement of this disconnected graph. For instance, $\mathfrak{S}_m[\mathfrak{S}_2]$, of order $m!2^m$, is the group of the graph consisting of m separate segments with their $2m$ ends, or of the complementary graph consisting of the vertices and edges of the cross polytope β_m, which is the octahedron when $m=3$ [see Coxeter, Regular Polytopes, Methuen, London, 1948; Pitman, New York, 1949, pp. 121, 133; these Rev. **10**, 261]. Again, $\mathfrak{S}_m[\mathfrak{S}_3]$, of order $m!6^m$, is the group of the graph consisting of m separate triangles, or of its complements, which is Thomsen's graph when $m=2$ [Blaschke and Bol, Geometrie der Gewebe, Springer, Berlin, 1938, p. 35] and Pappus's graph when $m=3$ [Kagno, Amer. J. Math. **69**, 859–862 (1947); these Rev. **9**, 196]. Finally, $\mathfrak{S}_m[\mathfrak{C}_n]$, of order $m!n^m$, is the group defined by the relations

$$S^{2m}=T^2=(ST)^{n(m-1)}=S^{-m}TS^mT=(S^{-1}TST)^2=(S^{-j}TS^jT)^2=1$$

$(j=2, 3, \cdots, [\frac{1}{2}m])$ [Coxeter, Proc. London Math. Soc. (2) **41**, 278–301 (1936), in particular, p. 286].

H. S. M. Coxeter (Toronto, Ont.).

Van Straten, Mary Petronia. The topology of the configurations of Desargues and Pappus. Rep. Math. Colloquium (2) **8**, 3–17 (1948).

Menger suggested [in lectures] the topological representation of projective configurations by graphs, a point of the graph representing a point of the configuration, and two points being joined by an arc of the graph if the corresponding points lie in a line of the configuration. The present problem is the embedding of such a graph in a surface so as to be the vertices and edges of a polygonal dissection of the surface. The theorems proved are as follows. (i) The graph representing the Desargues configuration cannot be embedded in a torus, but if any one arc is removed it can. (ii) That representing the Pappus configuration can be embedded in a torus, cutting it into eighteen triangles, but if any one further arc is added between two points not already joined it cannot. The proof of (i) is based on Kagno's theorem [J. Math. Physics **16**, 46–75 (1937), p. 52] of which an independent proof is given, that the graph consisting of two triads of points, each point of one triad

being joined by an arc to each point of the other (which is contained in the Desargues graph) can be embedded in a torus in just two ways, cutting it into one decagon and two quadrangles, or into three hexagons. Actual embeddings of the Desargues graph in a pretzel, and of this minus one arc in a torus are given. These results have been commented on and amplified by Coxeter [see the following review].

P. Du Val (Athens, Ga.).

Coxeter, H. S. M. Configurations and maps. Rep. Math. Colloquium (2) **8**, 18–38 (1949).

The paper falls into two distinct sections. The first, evidently intended as introduction (though in bulk almost half the paper) is a study of all possible regular polygonal divisions of the torus. These are either triangles, six at each vertex, squares, four at each vertex, or hexagons, three at each vertex. The possible numbers of vertices, edges, and faces depend on two integral parameters in each case. The symmetry groups of these figures are studied, generating operations specified, and relations found between these which constitute the abstract definition.

In the main section, a configuration of V points and aV/b lines, with a lines through each point and b points on each line (indicated by V_a in the special case $b=a$) is represented by a graph, with a vertex for each point of the configuration, points joined by a line in the configuration corresponding to vertices linked by an edge of the graph. The problem considered is to embed the graph in a surface, i.e., to identify its vertices and edges with those of a polygonal division of the surface, regular or as nearly so as possible. (The problem is not known to be always soluble, nor if a solution exists is it necessarily unique.) If the faces are l-gons the surface has characteristic $K=V\{1-a(b-1)(\frac{1}{2}-l^{-1})\}$.

The following cases are studied (configurations bracketed together give the same graph) together with a few simple cases given as examples:

7_3 Complete heptagon	14 triangles	$K=0$
8_3 Möbius tetrahedra	16 triangles	$K=0$
9_3 (Pappus)	18 triangles	$K=0$
Complete enneagon; Inflexions of plane cubic	24 triangles	$K=-3$
10_3 (Desargues); Complete pentagram	15 squares	$K=-5$
Complete decagon	30 triangles	$K=-5$

All but the second and last of these are illustrated.

Special attention is given to the figure of 15 squares whose vertices and edges are the graph of the Desargues configuration. It is realised in Euclidean space of four dimensions by the diametral squares of the five octahedra whose vertices are the midpoints of the edges of a regular simplex, which (as each side of each belongs to precisely one other) form a skew regular polyhedron. It was proved by Van Straten [see the preceding review] that this graph cannot be realised on a torus; a simplified proof is given. *P. Du Val.*

Tutte, W. T. On the imbedding of linear graphs in surfaces. Proc. London Math. Soc. (2) **51**, 474–483 (1949).

Starting with a given linear graph all of whose vertices are triple, the author considers all orientable surfaces on which the graph determines the boundary of a map. If the vertices of the graph admit of a two-coloration, he shows how to determine from this all surfaces whose maps have four-colorations. He also shows that for any connected graph there is one surface whose map admits of a three-coloration. *P. Franklin* (Cambridge, Mass.).

Calabi, Lorenzo. Topologia astratta. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 428–457 (1948).

A syllabus for a brief course in abstract spaces based upon the concepts of connectedness, compactness, regularity, normality, etc. A chart shows the inclusion relationships between different classes of spaces. *P. A. Smith.*

de Possel, René. Initiation à la topologie. Anais Fac. Ci. Pôrto **31**, 220–239 (1946).

Hewitt, Edwin. A class of topological spaces. Bull. Amer. Math. Soc. **55**, 421–426 (1949).

The author considers spaces admitting of a one-to-one continuous mapping onto a bicompact Hausdorff space. His main result concerns Q-spaces [Hewitt, Trans. Amer. Math. Soc. **64**, 45–99 (1948); these Rev. **10**, 126]. A Q-space possesses the above property if and only if the family $\mathfrak{Z}(X)$ (consisting of all $Z(f)$, where $Z(f)$ is the set of all $x \in X$, $f(x)=0$, and f is a real-valued continuous function in X) contains a subfamily $\mathfrak{A}$ such that (1) given $p \neq q$ in X, there is a set $B \in \mathfrak{A}$ containing exactly one of p and q; the intersection of all $B \in \mathfrak{A}$ is void; (2) any subfamily of $\mathfrak{A}$ with the finite intersection property has nonvoid total intersection.

M. Katětov (Prague).

Vaccaro, Michelangelo. Sulle matrici d'incidenza generali di un complesso topologico. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **7**, 368–372 (1948).

Let $E^a_{i_a}, \cdots, E^c_{i_c}$ be a nonordered collection of cells of dimensions $a, \cdots, c$ belonging to an abstract complex C. Let $\eta^{a;\cdots;c}_{i_a \cdots i_c}$ be 1 or 0 according as the cells are incident to each other in pairs or not. The author calls the hypermatrix $H^{a\cdots c}$ obtained by holding $a, \cdots, c$ fixed a generalized incidence matrix of C and makes a number of elementary observations about such matrices and their intersections and unions (appropriately defined). *P. A. Smith.*

Hirsch, Guy. À propos d'un problème de Hopf sur les représentations des variétés. Ann. of Math. (2) **50**, 174–179 (1949).

Das im Titel erwähnte Problem bezieht sich auf Variationen des Satzes von Borsuk: Eine antipodentreue Abbildung der n-dimensionalen Sphäre auf sich hat ungeraden Grad [cf. H. Hopf, Portugaliae Math. **4**, 129–139 (1944); diese Rev. **6**, 165]. Im Einverständnis mit dem Verfasser stellt der Referent fest, dass in dem Satz, der den Inhalt der vorliegenden Arbeit bildet, die Voraussetzungen abgeschwächt und die Behauptungen verstärkt werden können, ohne dass an dem Beweis etwas Nennenswertes zu ändern wäre; der Satz lautet dann: Es seien: M, M_1 n-dimensionale geschlossene orientierte Mannigfaltigkeiten mit Riemannscher Metrik; $n>1$; r_1 geodätischer Schlichtheitsradius von M_1 (d.h. zwischen je zwei Punkten, deren geodätische Entfernung $<r_1$ ist, existiert genau eine kürzeste Verbindung); e, e_1 die Eulerschen Charakteristiken von M, M_1; T eine stetige Abbildung von M in M_1 mit folgender Eigenschaft: es gibt eine positive Zahl v, sodass für jeden geodätischen Bogen der Länge v in M die Bilder seiner Endpunkte eine Entfernung haben, die >0 und $<r_1$ ist. Behauptungen: (1) der Grad c von T ist ungerade; (2) es gibt eine ungerade Zahl c', sodass $ec'=e_1c$ ist. Der Beweis beruht auf der Betrachtung der $(2n-1)$-dimensionalen Linienelement-Man-

nigfaltigkeiten $\bar{M}$, $\bar{M}_1$ von M, M_1 und gewisser Abbildungen von $\bar{M}$ in $\bar{M}_1$, die unter den Voraussetzungen des Satzes durch T bewirkt werden (die Voraussetzung "$n>1$" wird auf p. 178, Zeilen 9–11, benutzt). *H. Hopf* (Zürich).

Eckmann, Beno. Sur les applications d'un polyèdre dans un espace projectif complexe. C. R. Acad. Sci. Paris **228**, 1397–1399 (1949).

Let Y be a polytope i-simple for every i, and such that $\pi_q(Y)\neq 0$, $\pi_r(Y)\neq 0$ and $\pi_i(Y)=0$ for $1\leqq i<q$, and $q<i<r$. Let P be an arbitrary polytope, P_r its r-skeleton, and $c^{r+1}(f)\in H^{r+1}(P,\pi_r(Y))$ the cohomology class of the obstruction cocycle of $f:P_r\to Y$. Since the integral q-cycles of Y are a direct summand of the integral q-chains, there is a natural map $r:C_q(Y)\to H_q(Y)$ and so a $(q, H_q(Y))$-cocycle e^q can be defined in Y by setting $e^q(\sigma_q)=r(\sigma_q)$. Under the conditions stated on Y, (a) the cohomology class $[e^q]$ of e^q is uniquely defined, (b) given any $\alpha^q\in H^q(P, H_q(Y))$ there exists an $f:P_r\to Y$ with $f^+[e^q]=\alpha^q$ and (c) if $f, g:P_r\to Y$ and $f^+[e^q]=g^+[e^q]$, then $c^{r+1}(f)=c^{r+1}(g)$. Thus, setting $\phi[f^+e^q]=c^{r+1}(f)$, there is at hand a mapping

$$\phi:H^q(P, H_q(Y))\to H^{r+1}(P,\pi_r(Y)).$$

It is remarked that in case $Y=S^q$ and $r=q+1$, the mapping ϕ has been described by Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154]. If Y is the complex projective plane of $2m$ real dimensions, the hypotheses are satisfied for $q=2$ and $r=2m+1$, where $\pi_2(Y)=\pi_{2m+1}(Y)=$ integers. An easy argument shows that ϕ can then be given by the formula $\phi[f^+e^2]=c^{2m+2}(f)=[f^+e^2]^{m+1}$, the product being taken in the cohomology ring of P over the integers.

J. Dugundji (Los Angeles, Calif.).

Gordon, I. I. The classification of the mappings of a complex in a projective space. Doklady Akad. Nauk SSSR (N.S.) **65**, 441–444 (1949). (Russian)

Let K_n denote a given n-dimensional complex, and P_n projective n-space. The author studies the homotopy classes of mappings of K_n into P_n. His principal result is announced as follows. (1) In the case that n is odd and K_n is orientable, the homotopy classes are in natural one-to-one reciprocal correspondence with the degree of the covering of the mappings. If n is odd, but K_n is not orientable then there are only two homotopy classes. (2) In case n is even and K_n orientable there are two homotopy classes. If n is even and K_n is nonorientable then the homotopy classes are in one-to-one correspondence with the absolute degree of the mapping provided that a certain associated complex L_n is orientable. The case that this complex L_n is nonorientable (n even and K_n nonorientable) remains open.

Let F denote the fundamental group of K_n, and let f be a given map of K_n into P_n. Let $F_0(f)$ denote the subgroup of F of paths of K_n which are mapped by f into paths on P_n homotopic to zero. The following theorem is stated. Two maps f and g of K_n into P_n are homotopic on K_{n-1}, the $(n-1)$-dimensional skeleton of K_n, if and only if $F_0(f)$ and $F_0(g)$ coincide. To each subgroup F_0 of index two in F there corresponds a mapping f of K_n into P_n for which the associated $F_0(f)$ is F_0. Each F_0 of index two determines uniquely a two-sheeted covering L_n of K_n. This covering complex figures prominently in the theorem stated above, even where it is not explicitly mentioned. *L. Zippin.*

Whitehead, J. H. C. Note on a theorem due to Borsuk. Bull. Amer. Math. Soc. **54**, 1125–1132 (1948).

Consider a compactum A, a subcompactum B and a continuous mapping g of B into some compactum B' disjoint to A. Let A' denote the space obtained from $A\cup B'$ by identifying x with $g(x)$ for every $x\in B$. Theorem 1: A' is an ANR if A, B and B' are ANR's. The theorem [Borsuk, Fund. Math. **24**, 249–258 (1935)] to which reference is made in the title states that under these conditions A' is locally contractible. Theorem 1 is a nonempty extension in view of the recently constructed example [Borsuk, Fund. Math. **35**, 175–180 (1948); these Rev. **10**, 315] of an (infinite dimensional) locally contractible compactum which is not an ANR. Theorem 2: if B and B' belong to the same homotopy type and g is one of the mappings having a two-sided homotopy inverse then A' belongs to the same homotopy type as A. In the rest of the paper there is a discussion of the identification process, a sufficient condition for a compactum to be an ANR, and a new proof, using theorem 1, of part of a homotopy extension theorem due to Hu [Doklady Akad. Nauk SSSR (N.S.) **57**, 231–234 (1947); these Rev. **9**, 154]. *R. H. Fox.*

Whitehead, J. H. C. On the homotopy type of ANR's. Bull. Amer. Math. Soc. **54**, 1133–1145 (1948).

A connected compact metrizable ANR is called a CR-space. The symbol ΔX denotes the "quasi-dimensionality" min dim K, where K ranges over the set of those finite simplicial complexes which can be mapped into the given CR space X by means of mappings which have right homotopy inverses. Consider a mapping f of a CR-space X into a CR-space Y and denote max $(\Delta X, \Delta Y)$ by N. In order that f have a two-sided $(N-1)$-homotopy inverse (and consequently X and Y belong to the same $(N-1)$-homotopy type) it is necessary and sufficient that the induced homomorphisms $f_n:\pi_n(X)\to\pi_n(Y)$ be isomorphisms onto for $1\leqq n\leqq N-1$. If f_n are isomorphisms onto for $1\leqq n\leqq N$ then f has a two-sided homotopy inverse (and hence X and Y belong to the same homotopy type).

R. H. Fox (Princeton, N. J.).

Knobelauch, E. A. Extensions of homeomorphisms. Duke Math. J. **16**, 247–259 (1949).

The author obtains necessary and sufficient conditions that a homeomorphism ϕ of a Peano continuum M_1 into a Peano continuum M_2 be extendable to the manifolds (toruses or projective planes) K_1 and K_2 in which the continua are respectively imbedded. The methods used follow in general that of Adkisson and MacLane [same J. **6**, 216–228 (1940); these Rev. **1**, 221] where the containing spaces are planes or spheres. The notion of relative sense on pairs of triods (curves homeomorphic with the pattern **T**) plays an important role and in this connection the author defines and employs quite effectively index curves; i.e., curves homeomorphic with the patterns **H** and **X**. The following conditions that ϕ be extendable are obtained: (1) ϕ must preserve relative sense on every index curve in K_1 (stated here intuitively; the author states it more completely); (2) if C_1 is a simple closed curve of K_1, then C_1 bounds a 2-cell of M_1 if and only if $\phi(C_1)$ bounds a 2-cell of M_2; (3) if C_1 is a simple closed curve of K_1 which bounds a 2-cell S_1 of M_1 (this 2-cell is unique) and if $x_1\subset K_1-C_1$ then $x_1\subset S_1$ if and only if $\phi(x_1)\subset S_2$, where S_2 is the 2-cell of M_2 bounded by the simple closed curve $\phi(C_1)$. The author notes that (3) follows from (1) and (2) except in rather simple special cases. *V. W. Adkisson* (Fayetteville, Ark.).

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Belgodère, Paul. Les géométries de figures orientées. Gaz. Mat., Lisboa 10, no. 39, 2–4 (1949).

This is a discussion of geometrical orientation from the viewpoint of groups of transformations. *E. Lukacs.*

Wunderlich, Walter. Ein Spiegelproblem. Monatsh. Math. 53, 63–72 (1949).

The author proposes the following problem: to determine a curve C such that any ray emanating from a given point A will after n reflections pass through a given point B. He discusses this problem only for $n=2$ and the additional restriction that either $A=B$ or that both A and B are points at infinity. *E. Lukacs* (China Lake, Calif.).

Court, N. A. A special tetrahedron. Amer. Math. Monthly 56, 312–315 (1949).

Terracini, Alejandro. On the relations between the mutual distances of four points of a line. Math. Notae 8, 1–5 (1948). (Spanish)

La formule d'Euler $\overline{BC}\cdot\overline{AD}+\overline{CA}\cdot\overline{BD}+\overline{AB}\cdot\overline{CD}=0$, liant les distances mutuelles de quatre points d'une droite est du type $\sum f(\overline{BD},\overline{AD})=0$, f étant une fonction de deux variables ($f(x,y)=xy$), la sommation s'étendant aux termes obtenus en fixant D et permutant circulairement A, B, C. La formule de Chasles relative aux distances mutuelles de trois points A, B, C a la même structure $\sum\overline{BC}=0$ ($F(x,y)=x$). Plus généralement, s étant un entier positif, on a $\sum[(\overline{BC}+\overline{AD})^s+(-1)^{s-1}(\overline{BC}-\overline{AD})^s]=0$, équation du type précédent avec $f^{(s)}(x,y)=(x+y)^s+(-1)^{s+1}(x-y)^s$. L'auteur recherche le polynome $f(x,y)$ le plus général de degré s tel que, pour quatre points quelconques d'une droite on ait $\sum f(\overline{BC},\overline{AD})=0$, et montre que $f(x,y)$ a la forme $\sum_{k=1}^{s}c_kf^{(k)}(x,y)$, les c_k étant des constantes arbitraires.

Si $l=\overline{BC}$, $m=\overline{CA}$, $n=\overline{AB}$, $p=\overline{AD}$, $q=\overline{BD}$, $r=\overline{CD}$, on a $\delta_s=f^{(s)}(l,p)+f^{(s)}(m,q)+f^{(s)}(n,r)=0$. Réciproquement si six variables (toutes différentes de zéro) l, m, n, p, q, r vérifient toutes les δ_s (il suffit qu'il en soit ainsi pour $s=1,2,3$) il existe quatre points d'une droite tels que $l=\epsilon\overline{BC}$, $m=\epsilon\overline{CA}$, $n=\epsilon\overline{AB}$, $p=\overline{AD}$, $q=\overline{BD}$, $r=\overline{CD}$, ϵ pouvant avoir l'une des deux valeurs ±1. Les résultats précédents sont susceptibles d'une interprétation géométrique hyperspatiale dans l'espace S_8 aux coordonnées non homogènes l, m, n, p, q, r. *P. Vincensini* (Besançon).

Decuyper, Marcel. Composition des similitudes planes. Application aux quadrilatères complets. Bull. Soc. Math. France 76, 49–58 (1948).

J. Hadamard has considered the problem [Bull. Amer. Math. Soc. 50, 520–528 (1944); these Rev. 6, 99]: to construct a quadrilateral $Q(XYZT)$ similar to a given quadrilateral q so that the (consecutive) sides XY, YZ, $\cdots$ of Q shall pass through four given points K, L, M, N, with emphasis on the indeterminate case of its solution. The author takes up this problem with simpler means and extends the discussion of the indeterminate case by considering the three diagonals of the complete quadrilateral Q. A solution of the determinate case of the problem was given half a century ago [Jeffares, Educational Times 51, 223 (1898), question 13485]. *N. A. Court* (Norman, Okla.).

Maeda, Kazuhiko. On Loud's theorem. Sci. Rep. Tôhoku Imp. Univ., Ser. 1, 31, 70–72 (1942).

The author gives an alternative proof of the chain of theorems given by F. H. Loud [Trans. Amer. Math. Soc. 1, 323–338 (1900)] that are built on n oriented lines in a plane. Initially the oriented circles are drawn to touch every three; these circles have points of concurrence and these points of concurrence lie on circles, and so on successively in the chain. *T. W. Chaundy* (Oxford).

Menger, Karl. La géométrie axiomatique de l'espace projectif. C. R. Acad. Sci. Paris 228, 1273–1274 (1949).

Die gewöhnlichen geometrischen Axiome sind weder bezüglich der Incidenz- noch bezüglich der Existenzaussagen dual. Verfasser stellt mit den Grundbegriffen Punkt, Ebene, Gerade, Incidenz 8 Verknüpfungsaxiome auf, die dem Dualitätsprinzip genügen. Zusammen mit 2 weiteren, in sich dualen Existenzaxiomen bilden sie ein System unabhängiger Axiome, das in einem projectiven Raum erfüllt ist. *R. Moufang* (Frankfurt am Main).

Menger, Karl. Independent self-dual postulates in projective geometry. Rep. Math. Colloquium (2) 8, 81–87 (1948).

Für die ebene projective Geometrie lässt sich ein System von 4 unabhängigen Axiomen aufstellen, das dem Dualitätsprinzip genügt, und aus dem die gewöhnlichen ebenen Axiome folgen, die nicht dual sind. Hierbei spielt ausser dem Begriff Punkt, Gerade die Configuration von Punkt und Gerade in vereinigter Lage und die Configuration von 2 Punkten, auf deren Verbindung sich 2 Geraden schneiden, die Rolle der geometrischen Grundgebilde, mit deren Hilfe sich die Theoreme von Pascal und Desargues einfach formulieren lassen. Die Begründung der projectiven Geometrie des Raumes erfolgt aus 5 dualen Axiomen mit den Begriffen Punkt, Gerade, Ebene, Incidenz und Existenz. Die entarteten räumlichen Geometrien, deren Elemente einen Simplex bilden oder deren Elemente aus einem ebenen Punkt- und Geradenfeld und einem dazu dualen Gebilde oder aus einer Punktreihe und einem dazu dualen Gebilde bestehen, werden durch das 5. Axiom ausgeschlossen. *R. Moufang.*

Baker, H. F. The dual of a theorem proved by F. Morley. Bull. Calcutta Math. Soc. 40, 226–228 (1948).

This is the substance of a letter to the late C. V. Hanumanta Rao [cf. Bull. Calcutta Math. Soc. 37, 131–132 (1945); 38, 109–112 (1946); these Rev. 7, 390; 8, 337]. It contains a neat analytic proof of the following theorem. Let U, V', W, U', V, W' be six points on respective generators u, v', w, u', v, w' of a quadric cone in projective space. By Pascal's theorem, the three lines $v'w\cdot vw'$, $w'u\cdot wu'$, $u'v\cdot uv'$ lie in a plane ω. By interchanging v' and w', or w' and u', or u' and v', we derive three other "Pascal planes" λ, μ, ν. Then the three points in which the lines $(V'W\cdot\lambda)(VW'\cdot\lambda)$, $(W'U\cdot\mu)(WU'\cdot\mu)$, $(U'V\cdot\nu)(UV'\cdot\nu)$ meet ω are collinear. *H. S. M. Coxeter* (Toronto, Ont.).

Hall, Marshall, Jr. Correction to "Projective planes." Trans. Amer. Math. Soc. 65, 473–474 (1949).

The paper appeared in the same Trans. 54, 229–277 (1943); these Rev. 5, 72.

Lorent, H. Transformations de courbes planes. III. Anais Fac. Ci. Pôrto 31, 129–144 (1946).

For the first two parts cf. same Anais 26, 5–20, 65–83 (1941); these Rev. 9, 301.

Scott, T. A geometrical note on the binary cubic form. Proc. Edinburgh Math. Soc. (2) 8, 87–88 (1948).

The invariant properties of a binary cubic form were interpreted by Sturm [J. Reine Angew. Math. 86, 116–145 (1879), in particular, pp. 121–133] with reference to the geometry of a twisted cubic curve. This note adds a further result and, by projection, derives and extends a corresponding property of the rational plane cubic curve.

T. W. Chaundy (Oxford).

Convex Domains

***Aleksandrov, A. D. Vnutrennyaya Geometriya Vypuklyh Poverhnostei. [Intrinsic Geometry of Convex Surfaces].** OGIZ, Moscow-Leningrad, 1948. 387 pp.

Nothing can make a geometer happier than seeing geometry prevail in geometry. Since the present work is a decisive step in this direction the reviewer rarely enjoyed a book more. It will doubtless greatly influence the future development of "differential" geometry. It is true that the content of the book was largely known before through a series of articles by the author, but these precluded a full appreciation because they contained either no, or only very sketchy, proofs. In contrast, the book proceeds at a leisurely pace with much emphasis on motivation and complete proofs, except where additional material is reported.

Chapter I describes in detail the purpose and the main results of the book; it also gives a brief account of the classical theory. According to J. J. Stoker [Amer. J. Math. 62, 165–179 (1940); these Rev. 1, 168], the boundary of a convex set with interior points in E^3 (not E^3 itself) is either homeomorphic to a sphere, or to a plane, or it is a pair of parallel planes or an infinite convex cylinder. A complete convex surface is defined as a component of the boundary of a convex set with interior points in E^3 (that is, the complete boundary except in the case of two parallel planes), or a doubly counted plane convex domain. A convex surface S is a relatively open subset of a complete convex surface S' with the property that any two points of S can be connected in S by a shortest connection (geodesic arc) in S'. The second chapter discusses first well-known properties of arc-length of curves in metric spaces. It then turns to two-dimensional manifolds with an intrinsic metric (here called briefly metric surfaces), which means that the distance of two points equals the greatest lower bound of the lengths of the curves connecting them. Convex surfaces have the following property (1): If C and C' are shortest connections of A, B and A, B' respectively, and B' is an interior point of C, then $C \supset C'$. The consequences of (1) for general spaces are studied, and it is shown that a metric surface with property (1) can be triangulated into geodesic triangles of arbitrary small diameter.

Chapter III deals with characteristic properties of the intrinsic metric of convex surfaces. Since many of these results are obtained by approximation with polyhedra it is first shown that the intrinsic metric depends continuously on the surface: if the complete convex surfaces F_n tend to F and the points x_n, y_n on F_n tend to the points x, y on F, then the geodesic distances of x_n and y_n on F_n tend to the geodesic distance of x and y on F. Let C_1, C_2 be geodesic arcs on a convex surface S with the common initial point p, and x_i, y_i points on C_i such that y_i lies between x_i and p. If $p'x_1'x_2'$ and $p'y_1'y_2'$ are triangles in the Euclidean plane whose sides are equal to the corresponding sides of the geodesic triangles px_1x_2 and py_1y_2 on S, then $\angle x_1'p'x_2' \leqq \angle y_1'p'y_2'$. This property is first established for polyhedra and then by approximation for arbitrary convex surfaces. It is called the convexity condition and is fundamental because it is shown later that it characterizes convex surfaces. This condition implies among other things the above property (1), the existence of the angle (2) $\angle(C_1, C_2) = \lim \angle x_1'p'x_2'$, and that in any sufficiently small geodesic triangle on any convex surface (or in any geodesic triangle on a complete convex surface) the angles are at least as large as the angles in the plane triangle with the same sides. With a similar interpretation as above for the distances it is shown that the angle depends lower semicontinuously on the surface. The angle $\angle(C_1, C_2)$ exists also in the "strong sense": if x_1 tends to p on C_1 and x_2 moves so on C_2 that the shortest connections of x_1 and x_2 tend to a subarc of C_2, then $\angle x_1'p'x_2' \to \angle(C_1, C_2)$. (Only if also $x_2 \to p$ is this statement contained in the preceding one.)

The next chapter treats the concept of angle in greater detail. It is first shown that existence of angles in the sense of (2) implies additivity on any metric surface. Let C_1, C_2 be two geodesic arcs on a convex surface issuing from O. They divide a sufficiently small neighborhood of O into two parts which are called sectors. If U is one of these sectors, let $D_1 = C_1, D_2, \cdots, D_n = C_2$ be geodesic arcs issuing from O in U in the indicated order. Then $\sup \sum \angle(D_i, D_{i+1})$ is called the angle of the sector U. One of the two sector angles formed by C_1 and C_2 is equal to $\angle(C_1, C_2)$. The sum of the two sector angles is independent of the choice of C_1 and C_2 and does not exceed 2π. It is called the complete angle at O. A point whose complete angle is less than 2π is not an interior point of a geodesic arc. In addition to the previously mentioned lower semicontinuity of angle it is shown that, if the complete angle of F_n at O_n tends to the complete angle of F at O, then angles at O_n tend to the corresponding angle at O. A geodesic arc on a convex surface in E^3 has, as a curve in E^3, everywhere a right-hand and a left-hand tangent. If C_1 and C_2 are two geodesic arcs issuing from O and U, V the corresponding sectors, moreover T the tangent cone of the surface at O, then the angles on T (flattened out) formed by the tangents of C_1 and C_2 are equal to the angles of the sectors defined above.

The subject of chapter V is integral curvature. An elementary set consists of disjoint open geodesic triangles, open geodesic arcs and points. The integral curvature of a triangle is defined as the excess, the integral curvature of a geodesic arc is 0, and that of a point is 2π minus the complete angle at the point. By addition the integral curvature of polygonal regions is defined. It is independent of the triangulation, because the usual relation between integral curvature, characteristic, and the angles at the boundary is shown to hold. This integral curvature is then extended to a completely additive set function on all Borel sets. The extrinsic definition of integral curvature on a convex surface S is obtained by associating with any Borel set M on S the measure of the set of points on the unit sphere at which (oriented) tangent planes parallel to the (oriented) supporting planes of S at points of M exist. The main result of the chapter is that Gauss's theorem holds under these general

conditions, that is, that the two definitions of integral curvature coincide for Borel sets.

A convex polyhedron is defined intrinsically as a metric surface S such that every point p of S has a neighborhood which is isometric to a neighborhood of the apex of a cone with complete angle not exceeding 2π. The main content of chapter VI is the proof that if S is homeomorphic to a sphere and intrinsically a convex polyhedron, one, and except for motions only one, convex polyhedron in E^3 exists which is isometric to S. The proof proceeds by induction with respect to the number of vertices, that is, those points whose complete angle is less than 2π.

Chapters VII and VIII are the central part of the book. Chapter VII is devoted to the proof of (3): A metric surface which is homeomorphic to a sphere and satisfies the convexity condition is isometric to a convex surface in E^3. Actually an apparently much weaker condition than the convexity condition is used. The angle in the strong sense defined above need not exist on an arbitrary metric surface, but the corresponding inferior limit exists, of course, always. The excess condition, which replaces the convexity condition, requires that the excess of small geodesic triangles formed with these inferior limits is nonnegative. Although (3) does not contain the theorem of Weyl-Lewy on the realization of a given analytic line element with positive curvature on a sphere by an analytic convex surface in E^3, inasmuch as the analytic character of the resulting surface does not follow, (3) is doubtless the final theorem in this direction from the geometric point of view. Also, (3) seems entirely out of reach of analysis. The idea of the proof is to triangulate the surface, replace each triangle by a plane triangle with sides of the same lengths, show that the resulting abstract surface is intrinsically a convex polyhedron, apply the theorem on the realization of polyhedra, and pass to the limit under refinement of the triangulation.

If the manifold S originates from geodesic polygonal regions which satisfy the convexity condition by piecing them together and is such that the complete angles at all vertices do not exceed 2π, then S itself satisfies the convexity condition. With the help of this theorem it is shown that a plane with a finitely compact intrinsic metric and the excess or convexity condition is isometric to a complete (unbounded) convex surface in E^3, a result which nobody even attempted to obtain with analytical methods. Any point on a metric surface which satisfies the excess condition has a neighborhood which is homeomorphic to a circular disc and isometric to a convex surface. The proof for the following general theorem is only outlined: a metric surface is isometric to a convex surface if and only if (1) any two points have a shortest connection, (2) every point has a neighborhood U such that every geodesic triangle in U satisfies the excess condition, (3) the surface is homeomorphic to a subset of a sphere.

If the curve C on a convex surface has tangents at its end points a, b, and a, b are connected by a geodesic polygon P on the right side of C which forms the angles α, β with C at a and b and has angles δ_i at its vertices measured on the side away from C, then $\alpha+\beta+\sum(\pi-\delta_i)$ tends to a limit L as P approaches C from the right. If curve and surface are sufficiently smooth, L is the integral over the geodesic curvature of C properly oriented. Call L the right integral geodesic curvature (i.g.c.) of C. The Gauss-Bonnet formula holds for this i.g.c. and the above defined integral curvature of the surface. A domain D on a complete convex surface is convex if it contains with any two points a shortest connection (on the surface). An arc on the boundary of D has nonnegative i.g.c. towards D. If D is compact then it is either a complete (closed) convex surface, or homeomorphic to a circular disc or isometric to a zone on a circular cylinder. In the second case it is isometric to a cap on a convex surface, that is, a domain bounded by a plane curve and such that its projection on the plane of the curve coincides with plane domain bounded by the curve. If every subarc of a curve has nonnegative i.g.c. towards both sides (for instance the edge of a lens) then the curve is called a quasi-geodesic. Whereas there need not exist a geodesic arc issuing from a given point in a given direction, even if the surface has a tangent plane at that point, there is always a quasi-geodesic through a given line element. Each closed convex surface contains at least one simple closed quasi-geodesic. Some properties and peculiarities of geodesic circles are studied.

The following chapter X is concerned with area. The intrinsic area is defined by triangulating the surface, adding the areas of the corresponding Euclidean triangles with the same sides, and passing to the limit by refining the triangulation. It is shown that the area thus obtained coincides for Borel sets with the usual extrinsic area. Certain variational problems are solved, for instance: among all convex surfaces homeomorphic to a circle and with a given perimeter and integral curvature less than 2π the lateral surface of a right circular cone has the greatest area.

Convex surfaces where the ratio of the integral curvature $\alpha(G)$ to the area $A(G)$ is bounded for any subdomain G are studied in chapter XI. For every point p there exists a number $\rho(p)>0$ such that a geodesic arc of length $\rho(p)$ beginning at p with a given direction exists. It follows from property (1) that there is only one such arc through a given point $q\neq p$ with distance less than $\rho(p)$ from p. Hence polar coordinates can be introduced in the geodesic circle with radius $\rho(p)$ about p. The Gauss curvature $K(x)$ is said to exist at x if $\lim \alpha(G)/A(G)=K(x)$ for an open set G tending to x. If the Gauss curvature $K(r,\beta)$ exists in a geodesic circle U with polar coordinates r, β, then $ds^2=dr^2+G(r,\beta)d\beta^2$ and $K(r,\beta)=-G(r,\beta)^{-\frac{1}{2}}\partial^2 G(r,\beta)^{\frac{1}{2}}/\partial r^2$. If $\alpha(G)/A(G)$ is constant then the surface is isometric to a subset of a sphere. If $\alpha(G)/A(G)>K>0$ or $\alpha(G)/A(G)<K$, then the surface can be compared to a sphere with radius $K^{-\frac{1}{2}}$. For instance, in the first case the angles in a geodesic triangle on the surface are greater than the angles in the spherical triangle with the same sides. If the surface is imbedded in E^3 and $\delta<\alpha(G)/A(G)<\epsilon$, then the surface has a continuous tangent plane and is strictly convex. Therefore the above theorem (3) actually yields a realization of the given line element (and not only of the finite distances) and the given Gauss curvature if interpreted in the present sense. The interpretation as $R_1^{-1}R_2^{-1}$ is in general not possible because surfaces with a continuous Gauss curvature in the present sense exist, where R_1 and R_2 do not exist (at single points; they exist almost everywhere on any convex surface).

The final chapter sketches the extension of these results to surfaces in spherical and hyperbolic spaces. An appendix furnishes the elementary theorems on convex bodies used in the text. The content of the book is almost entirely due to the author and his students. To a minor extent the reviewer and Feller's work on the (extrinsic) differential geometry of convex surfaces and Cohn-Vossen's work on geodesics and integral curvature are used.

H. Busemann (Los Angeles, Calif.).

Klee, V. L., Jr. A characterization of convex sets. Amer. Math. Monthly 56, 247–249 (1949).

The author proves that a closed subset S of a finite-dimensional Euclidean space with interior points is convex if and only if any foot q on S of an arbitrary exterior point p (i.e., a point q of S nearest to p) is a foot on S of an unbounded set of points. *L. Fejes Tóth* (Budapest).

Mirguet, Jean. Sur l'équivalence de la double courbure et de la non-convexité. C. R. Acad. Sci. Paris 228, 1474–1476 (1949).

The author proves that every point of an orthosurface with finite upper paratingent which is not the limit of points with nondegenerate double curvature is interior to a domain which is convex in the wide sense (consists of points of support). *H. L. Smith* (Baton Rouge, La.).

Buck, R. C., and Buck, Ellen F. Equipartition of convex sets. Math. Mag. 22, 195–198 (1949).

It is shown that every bounded plane convex set may be divided by three concurrent straight lines into six regions of equal area, but no such convex set may be divided into seven regions of equal area by three nonconcurrent straight lines. It is also shown that if six of the seven regions into which a plane convex set is divided by three nonconcurrent lines have equal area, then the seventh unequal region must be the central triangular piece; and it is conjectured that the area of this central region can be at most one-eighth the area of the equal regions. *O. Frink* (State College, Pa.).

Algebraic Geometry

*Weil, André. **Variétés abéliennes et courbes algébriques.** Actualités Sci. Ind., no. 1064 = Publ. Inst. Math. Univ. Strasbourg 8 (1946). Hermann & Cie., Paris, 1948. 165 pp.

This book is a continuation of the author's test of the comprehensive methods developed in his "Foundations of Algebraic Geometry" [Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. 9, 303], which was begun in his "Sur les courbes algébriques et les variétés qui s'en déduisent" [Actualités Sci. Ind., no. 1041 = Publ. Inst. Math. Univ. Strasbourg 7 (1945); these Rev. 10, 262] with the proof of the Riemann hypothesis for function fields of one variable. Now the theory of Abelian functions and of the Jacobian variety attached to a curve is developed over arbitrary coefficient fields. All the classical results (i.e., one works over the field of all complex numbers) are generalized and new theorems enriching the classical theory are added. An essential tool, though this at first glance may appear to be trivial, is the concept of function defined on a variety U [abstract in the sense of "Foundations . . ." pp. 167 ff.] with values in another variety V. This is naturally done by means of special [regularity in the sense of "Foundations . . ." p. 107] subvarieties of the Cartesian product $U \times V$; it is significant that heretofore a systematic exploitation of this particular concept was not used appreciably in modern algebra. Thus the type of reasoning of analysis and combinatorial topology can now find fuller application in arithmetic and algebraic geometry.

Already, for the proof of the Riemann hypothesis, it was necessary to consider Cartesian products of a curve involving more than two factors [theorem 10 of "Sur les courbes . . ."]; only in this fashion, it appears, the full force of algebraic geometry can be brought to bear upon certain delicate parts of the class field theory of function fields; this involves a detailed study of the ring of correspondences [e.g., the positiveness of certain traces, and L-series in § 5 of "Sur les courbes . . ."]. The full details of the pertinent facts are given in this book by means of a systematic theory of Abelian varieties, i.e., complete ["Foundations . . ." p. 168] algebraic varieties which are group spaces in §§ II, III and IV (see the beginning of this review for the possibility of this definition). A basic result for the theory of Abelian varieties is embodied in theorem 9 on page 34 which states that a function on an Abelian variety B into an Abelian variety A is (essentially) a homomorphism in the sense of the given group operations on the varieties. (This result is a "dual" of the known direct imbedding of differentials.) In § V the existence of Abelian varieties is established by means of the Jacobian varieties of curves. It is noteworthy that, as in the classical theory, the theory of curves and their attached Jacobian varieties is material in order to enrich the theory of general Abelian varieties [see the useful theorem 10 on p. 39, and later results starting with theorem 22 on p. 79 to theorem 24 on p. 87 leading up to important facts concerning the anti-isomorphism of the ring of endomorphisms].

To a homomorphism λ of an Abelian variety A into an Abelian variety B the norm $\nu(\lambda)$ is associated by means of the theory of cycles and projections, and this norm is investigated in relation to the group of points $x \in A$ which are mapped upon $0 \in B$ by means of λ [in theorem 12 with corollary and proposition 9, pp. 42, 43]. In particular, it is shown that $\nu(\lambda) = 0$ implies the existence of an Abelian subvariety of A which does not reduce to a point. Results of this kind are continued subsequently in the generalization of Poincaré's famous result on the reducibility of integrals of the first kind in theorem 26 on page 94 (exploitation of $\nu(a) > 0$, a an integer, in § VII, also important later for the proof of semi-simplicity of the algebra of homomorphisms of an Abelian variety), with the subsequent work on categories of Abelian varieties (defined by means of isogeny, i.e., equality of dimension and existence of homomorphisms upon each other). Thus the algebraic structure of the algebra of homomorphisms (ring of homomorphisms of an Abelian variety into itself extended by the field of rational numbers) is related to the decomposition of categories [theorem 29 on p. 101 in § VII]. These results comprehend incidentally a first step in the direction of an algebraic structure theory of certain function fields as subfields of direct products of function fields with special properties such as that of having multiplication algebras which are division algebras, in other words, here are the roots of an algebraic theory of "fibered" fields. The preparations for results of this kind are not easy, and intermediate statements using the theorem of Riemann-Roch have to be established. We mention theorem 19 on page 70 which concerns itself with the canonical mapping of the classes of divisors of degree 0 on a curve onto the group of points on the attached Jacobian variety, and theorem 21 on page 77 where connections between homomorphisms of Jacobian varieties and algebraic correspondences of the related curves are made precise. As a consequence an important part of an algebraic analogue of Hurwitz' transcendental description of algebraic correspondences is obtained.

Already the qualitative study of the homomorphisms λ of Abelian varieties leads in theorem 14 on page 49 to a

representation $M_l(\lambda)$ of λ by matrices with elements in the additive l-adic group modulo l, where l is relatively prime to the characteristic of the underlying coefficient field as is shown subsequently (the representation for the characteristic is of no special interest in the present work). In order to determine the precise degree of the matrices, as is done in theorem 23 and its corollaries 1 and 2 on pages 126–127, one has to pass through the delicate § VIII where the difficult work of theorem 23 in § VII is resumed. The degree in question is uniformly $2n$ if n is the dimension of the variety; in other words de Jonquières' formula is obtained. Thus the author succeeds in finding $2n$-dimensional representations $M_l(\lambda)$ of the ring of homomorphisms of an Abelian variety whose traces and norms coincide [theorem 36 on p. 136, passing through the theorem on Jacobian varieties in theorem 34 on p. 131] with the trace and norm which were previously defined without recourse to matrix representations [e.g., section 27 on p. 43 for the norm]. With this result the climax of the investigation is reached. Now a rather complete analogue has been found for Hurwitz's transcendental description of correspondences, and the proof of Artin's conjecture on the L-series of non-principal characters is completed [p. 137 and § V, pp. 72–85 in "Sur les courbes . . ."].

Finally in § X the famous result on the "finiteness of the base" of the ring of homomorphisms over the ring of integers is proved in theorem 37 on page 138. The proof is first given for simple Abelian varieties (no proper Abelian subvarieties are present, or equivalently the ring of homomorphisms lies in a division algebra); the theorem 31 on page 117 on the positiveness of certain traces is essential in order to set up a useful symmetric bilinear form on homomorphisms [p. 139]. The proof is then completed by a straightforward application of the theory of categories [theorem 29 on p. 101], and results of the theory of Riemann matrices become applicable [theorem 32 on p. 144], where naturally the originally important matrices of periods do not occur.

In the terminating § XI it is explained how one can map divisors of an n-dimensional Abelian variety upon $2n$-dimensional l-adic matrices where the mapping is a homomorphism acting upon classes in the sense of linear equivalence ["Foundations . . . ," chapter VIII]. These results offer extensions of facts, up to now restricted to the application of combinatorial topology to algebraic geometry, dealing with the arithmetic theory of homomorphisms [proposition 33 on p. 160, a kind of decomposition theory of ideals], and necessary and sufficient conditions (involving factor sets) for a divisor to be linearly equivalent to a multiple of another divisor [proposition 33 on p. 163].

O. F. G. Schilling (Chicago, Ill.).

van der Waerden, B. L. Über einfache Punkte von algebraischen Mannigfaltigkeiten. Math. Z. **51**, 497–501 (1948).

En se servant de la "zugeordnete Form" [Chow et van der Waerden, Math. Ann. **113**, 692–704 (1937)] d'une variété irréductible M, et d'une généralisation aux anneaux de séries formelles du lemme de Hensel, l'auteur montre que les coordonnées d'un point générique de M s'expriment par des séries formelles en fonction d'uniformisantes locales $(t_1, \cdots, t_d)$; d étant la dimension de M) en un point simple P de M; ces coordonnées sont des fonctions algébriques séparables des (t_i), et leurs dérivées partielles par rapport aux (t_i) peuvent s'exprimer par des fractions dont les dénominateurs ne s'annulent pas en P. En utilisant le critère de multiplicité un des intersections de variétés algébriques, il montre ensuite que la définition classique des points simples (par les matrices jacobiennes) coïncide avec celle de Zariski (l'anneau des quotients du point est un anneau local régulier) [Ann. of Math. (2) **40**, 639–689 (1939); ces Rev. **1**, 26]; le corps de base est supposé parfait, ce qui élimine toutes les difficultés [cf. Zariski, Trans. Amer. Math. Soc. **62**, 1–52 (1947); ces Rev. **9**, 99]. La généralisation du lemme de Hensel est due à Abellanas [Revista Acad. Ci. Madrid **36**, 482–499 (1942); ces Rev. **9**, 373] (et aussi à I. S. Cohen, dans le cas encore plus général des anneaux locaux complets [Trans. Amer. Math. Soc. **59**, 54–106 (1945); ces Rev. **7**, 509]). Le résultat relatif à la définition des points simples, avec essentiellement la même démonstration, se trouve chez C. Chevalley [Trans. Amer. Math. Soc. **57**, 1–85 (1945); ces Rev. **7**, 26]. *P. Samuel* (Paris).

van der Waerden, B. L. Birationale Transformation von linearen Scharen auf algebraischen Mannigfaltigkeiten. Math. Z. **51**, 502–523 (1948).

L'auteur s'occupe des transformations birationelles de systèmes linéaires de diviseurs, notamment du fait que le transformé birationel d'un système complet n'est pas toujours complet; ce fait était évité, chez les géomètres italiens, par l'introduction du concept de système linéaire avec points bases assignés; mais, au delà des surfaces, la notion de points infiniment voisins devient impraticable, et l'auteur la remplace par celle de valuation [cf. Zariski, Trans. Amer. Math. Soc. **53**, 490–542 (1943); ces Rev. **5**, 11]. L'article commence par quelques résultats d'utilité technique: si un point P de la variété non singulière M ne se trouve pas sur le diviseur des poles de la fonction φ, on peut écrire $\varphi = f/g$ où g n'est pas nul en P; il en est de même des dérivées de φ par rapport à des uniformisantes locales en P. Ces résultats sont immédiatement utilisés pour montrer l'invariance birationelle des tenseurs partout finis de Kähler [Accad. Ital. Mem. Cl. Sci. Fis. Mat. Nat. 3_1, 33–47 (1932)].

Sont ensuite considérées les valuations discrètes du corps des fonctions rationelles sur M^n dont le corps résiduel est de dimension algébrique $n-1$ (les "diviseurs premiers" de M); le centre N d'une telle valuation est défini [cf. Zariski, ibid.]; selon que N est de dimension $n-1$ ou $<n-1$, le diviseur premier est dit de première ou de seconde espèce; les éléments du groupe abélien libre engendré par les diviseurs premiers de première espèce sont appelés les diviseurs de M; en considérant (localement) un diviseur comme une intersection complète de M avec un diviseur de l'espace, la notion de valuation est étendue aux diviseurs. Ceci étant, on entendra par système linéaire, l'ensemble d'un système linéaire (au sens classique) S, et d'une famille de valeurs $(v(\mathfrak{p}_i))$ ($\mathfrak{p}_i$: diviseurs premiers de seconde espèce; $v(\mathfrak{p}_i)$: entiers tels que, en $\mathfrak{p}_i$, tout diviseur de S ait au moins la valuation $v(\mathfrak{p}_i)$); les $v(\mathfrak{p}_i)$ peuvent être <0; la notion de système complet est définie par rapport à cette notion. Pout tout diviseur premier $\mathfrak{p}$, soit $e(\mathfrak{p})$ le minimum des valuations (en $\mathfrak{p}$) des diviseurs de S; $e(\mathfrak{p})-v(\mathfrak{p})$ est un entier $\geqq 0$ appelé l'excès du système en $\mathfrak{p}$; si $\mathfrak{p}$ est de première espèce et si $e(\mathfrak{p})-v(\mathfrak{p})>0$, ceci donne, avec leurs multiplicités, les composantes fixes de S. Le système S' transformé birationel de S est défini par la condition de conservation des excès: $e(\mathfrak{p})-v(\mathfrak{p})=e(\mathfrak{p}')-v(\mathfrak{p}')$, $\mathfrak{p}'$ étant le diviseur premier correspondant à $\mathfrak{p}$; ceci détermine S' de façon unique; les notions de système complet et d'inclusion d'un système dans un autre sont conservées par une telle transformation. L'auteur définit ensuite les produits et

quotients de systèmes linéaires; tout système complet est le quotient de deux systèmes complets sans excès. Les systèmes complets forment un semi groupe; on complète celui ci en un groupe, dont les éléments sont appelés les systèmes complets virtuels; on peut définir ceux ci comme ensembles d'une classe de diviseurs linéairement équivalents, et d'une famille de valeurs $(v(\mathfrak{p}_i))$; la notion de système complet virtuel, contrairement à celle de classe de diviseurs, est un invariant birationel. *P. Samuel* (Paris).

Jongmans, F., et Nollet, Louis. Classification géométrique des faisceaux de courbes algébriques planes de genre deux. Bull. Sci. Math. (2) **72**, 80–96 (1948).

Die von den Verfassern durchgeführte Klassifikation der ebenen Büschel von Kurven des Geschlechts 2 bezüglich ihrer Cremonainvarianz ist vorher von de Franchis bereits unternommen worden [siehe Rend. Circ. Mat. Palermo **13**, 1–27 (1899)]. Es ergaben sich dort 7 arithmetisch verschiedene Typen von Kurvenbüscheln 4. bis 15. Grades, die durch ihr Basisverhalten zu beschreiben waren. Da diese Klassifikation bei de Franchis jedoch nach genauerer Untersuchung eine Reihe von Mängeln und Ungenauigkeiten aufweist, wird die ganze Frage hier erneut aufgegriffen. Die von de Franchis bereits richtig festgestellten 7 arithmetisch möglichen Typen werden bezüglich ihrer geometrischen Existenz untersucht, wobei der letzte Typ von Kurven 15. Grades sich wie dort als unmöglich erweist, während die anderen Fälle sich zum Teil in mehrere Möglichkeiten aufspalten, sodass sich 9 verschiedene, am Schluss noch besonders zusammengestellte Büscheltypen ergeben. *W. Burau* (Hamburg).

Jongmans, F., et Nollet, L. La classification des systèmes linéaires de courbes algébriques planes de genre quatre. Ann. Sci. École Norm. Sup. (3) **65**, 139–188 (1948).

Die Verfasser führen die umfangreiche Aufgabe durch, die Linearsysteme $|C|$ von Kurven des Geschlechts 4 bezüglich ihrer Cremonainvarianz zu klassifizieren. Die Untersuchung, die mit ausserordentlicher Sorgfalt vorgenommen wird, unterscheidet die Systeme, wie stets bei Cremonaklassifikationen, nach dem Verhalten ihrer Adjungierten. Im ersten Kapitel werden die Fälle behandelt, wo das erste adjungierte System $|C_1|$ einfach ist und das Höchstgeschlecht 3 besitzt, wobei noch zu unterscheiden ist, ob es aus hyperelliptischen Kurven besteht oder nicht. Im zweiten Kapitel wird der Fall behandelt, dass $|C_1|$ mit einer Involution oder einem Büschel zusammengesetzt ist. In diesen Fällen besitzen die Kurven des Systems $|C_1|$ das Geschlecht 2 oder 3. Benutzt werden dabei die Klassifikationsergebnisse für niederes Geschlecht, insbesondere die Resultate einer früheren Arbeit des einen Verfassers [Nollet, Mém. Soc. Roy. Sci. Liége (4) **7**, 469–554 (1947); diese Rev. **9**, 461].

Im Kapitel III werden dann diejenigen Linearsysteme untersucht, deren adjungiertes System $|C_1|$ ein Geschlecht $p_1 \geqq 4$ besitzt (es kann bis zu 5 ansteigen). Sämtliche erhaltenen Typen werden zum Schluss in einer fast 4 Seiten langen Aufstellung sehr übersichtlich zusammengestellt, wobei auch die arithmetischen Daten der weiteren adjungierten Systeme, soweit sie vorhanden sind, angegeben sind. *W. Burau* (Hamburg).

Néron, André. Un théorème sur le rang des courbes algébriques dans les corps de degré de transcendance fini. C. R. Acad. Sci. Paris **228**, 1087–1089 (1949).

Let K be any extension of finite transcendence degree of the rational field. The author states that, given any algebraic curve Γ of genus $p \geqq 1$ and finite rank r over K, there is an infinity of fields K^σ homomorphic to K and such that the homomorphism between K and K^σ transforms Γ into a curve Γ^σ having rank greater than or equal to r over K^σ.

Next he indicates three consequences of this theorem, extending one of his previous results [same C. R. **226**, 1781–1783 (1948); these Rev. **10**, 60], i.e., that, denoting by k an arbitrary finite algebraic field, (a) any pencil of plane elliptic cubics over k, no curve of which is reducible, contains some curves of rank greater than or equal to 8 over k; (b) there are some elliptic cubics of rank greater than or equal to 10 over k; (c) there are some curves of any given genus $p \geqq 2$, having rank greater than or equal to $3p+5$ over k. *B. Segre* (Bologna).

d'Orgeval, B. La propriété caractéristique des surfaces réglées. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 772–778 (1948).

Nouvelle démonstration de la suivante proprieté caractéristique des surfaces réglées, due à F. Enriques et G. Castelnuovo [Ann. Mat. Pura Appl. (3) **6**, 165–225 (1901)]: si sur une surface algébrique il existe un système linéaire de courbes, dont l'ordre n et le genre p vérifient la relation $n > 2p-2$, la surface est birationnellement équivalente a une surface réglée. *G. Zappa* (Naples).

Rossier, Paul. Sur les quartiques gauches. II. Arch. Sci. Soc. Phys. Hist. Nat. Genève **2**, 174–176 (1949).

For part I cf. same Arch. **1**, 503–504 (1948); these Rev. **10**, 320.

Godeaux, Lucien. Sur un procédé de construction de transformations birationnelles hyperspatiales. Časopis Pěst. Mat. Fys. **73**, 121–130 (1949). (French. Czech summary)

Soient $S_r(x_0, \cdots, x_r)$ et $S_s(y_0, \cdots, y_s)$ deux espaces indépendants dans un espace Σ à $r+s+1$ dimensions; soient analoguement $S_r'(x_0', \cdots, x_r')$ et $S_s'(y_0', \cdots, y_s')$ deux espaces indépendants dans un autre espace Σ' à $r+s+1$ dimensions. Si l'on donne entre S_r, S_r' et entre S_s, S_s' deux transformations birationnelles arbitraires:

$$x_0' : x_1' : \cdots : x_r' = \varphi_0 : \varphi_1 : \cdots : \varphi_r;$$
$$y_0' : y_1' : \cdots : y_s' = \psi_0 : \psi_1 : \cdots : \psi_s,$$

où les φ_i, ψ_i sont des formes algébriques des dégrés m, n en $x_0, \cdots, x_r$ et $y_0, \cdots, y_s$, on en déduit entre Σ, Σ' la transformation birationnelle:

$$x_0' : \cdots : x_r' = y_0' : \cdots : y_s' = \varphi_n\beta : \cdots : \varphi_n\beta : \psi_n\alpha : \cdots : \psi_n\alpha,$$

où $\alpha(x_0, \cdots, x_r)$ et $\beta(y_0, \cdots, y_s)$ sont des formes algébriques arbitraires, dont les dégrés λ, μ peuvent avoir les valeurs $n-1$, $m-1$ ou les valeurs $n+1$, $m+1$. Après ces considérations générales, l'auteur considère en particulier les cas $r=s=m=n=1$; $r=2$, $s=1$, $m=2$, $n=1$; et surtout le cas $m=n=1$, $\lambda=\mu=2$, avec des valeurs quelconques de r, s. Dans ce dernier cas on obtient une transformation cubique qui est une généralisation d'une transformation cubique bien connue de l'espace ordinaire; l'auteur en donne les propriétés principales. *E. G. Togliatti* (Gênes).

Differential Geometry

Busemann, Herbert. Spaces with non-positive curvature. Acta Math. **80**, 259–310 (1948).

The author's object is to study the geometry of spaces of nonpositive curvature along the lines of Hadamard's well-

known work [J. Math. Pures Appl. (5) 4, 27–73 (1898)], and of more recent work of H. Hopf, É. Cartan, S. Cohn-Vossen, J. Nielsen, and others. However, it is important to state at once that the author carries out his geometrical studies, in contrast to those of the other authors named, without introducing differentiability hypotheses and hence without the use of a Riemannian metric. It should also be added that these strong generalizations are accomplished with far less complication in the proofs than might have been expected.

The spaces R treated by the author are finitely compact metric spaces (also called complete spaces by some writers). They are convex in the following sense: if x and z are points of R, there should exist a point y such that the distances between the pairs of points satisfy the relation $xy+yz=xz$. Finally it is assumed that prolongations are possible and unique in the small. This means that to every point of R a neighborhood U exists such that if x and y belong to U, then a point z exists in R such that $xy+yz=xz$, and the resulting prolongation is unique in an obvious sense. In such a space the geodetic connection is locally unique.

The notion of nonpositive curvature for a space R is defined as follows. To every point of R a neighborhood U is assumed to exist such that the side bc of any geodetic triangle abc in U is at least twice as long as the shortest geodetic arc connecting the midpoints b', c' of the other two sides: (*) $bc \geqq 2b'c'$. The author shows that the condition (*) is equivalent to the condition of nonpositive curvature in case the metric is Riemannian in character. He shows that the condition (*) is all that is needed to extend the theory of spaces with nonpositive Riemannian curvature to the much more general spaces of the type R. The importance of the condition (*) lies in the fact that it can be made to serve as a replacement for the Gauss-Bonnet formula for such purposes as, for example, to show the nonexistence of geodetic "Einecke" and closed geodesics which are homotopic to 0. A complete theory of parallels in such spaces then becomes possible.

A few of the author's results may be summarized as follows. It is shown (under the additional assumption that the spaces R have the property of domain invariance) that the universal covering space of an R-space with nonpositive curvature is a straight line space, i.e., that any two points of the covering space are on at most one geodesic and also that every geodesic is congruent (in a rather obvious sense) to the Euclidean straight line. In particular, a simply connected space R (with domain invariance) and nonpositive curvature is a straight line space. Thus the study of general spaces with nonpositive curvature is reduced to studying simply connected straight line spaces by means of the covering motions in the universal covering space. The application of these facts leads to the two results on which Hadamard's work is primarily based: in a space with nonpositive curvature there is only one geodetic arc within a given homotopy class connecting two given points, and every free homotopy class contains at most one closed geodesic. The author then goes on to study certain special spaces. One such space is the space with zero curvature (i.e., a space in which the equality sign in (*) holds everywhere); the author shows such spaces to have finite connectivity. He also shows that any torus with nonpositive curvature necessarily has zero curvature everywhere. Another special class of spaces considered is the class of compact spaces with negative curvature; one result obtained for them is that such spaces cannot have an Abelian fundamental group. The author takes up the theory of two-dimensional manifolds and shows that classic results of F. Klein regarding the topological types of manifolds which can be provided with Riemannian metrics of a given constant curvature can be extended to the spaces R having curvature of constant sign. For example, he shows that all two-dimensional manifolds except the sphere and the projective plane can be provided with metrics having nonpositive curvature. The author also studies compact manifolds of negative curvature, with results of the same type as those obtained recently by Nielsen for surfaces of constant negative curvature. *J. J. Stoker.*

Lyusternik, L., and Šnirel'man, L. Topological methods in variational problems and their application to the differential geometry of surfaces. Uspehi Matem. Nauk (N.S.) 2, no. 1(17), 166–217 (1947). (Russian)

This article covers, with one exception which is discussed later, the same material as an earlier monograph [Méthodes topologiques dans les problèmes variationnels, Actual. Sci. Ind., no. 88, Hermann, Paris, 1934]. The difference between the two articles is that the present work fills in gaps and amplifies proofs of the former work, which according to the authors, proved to be too condensed. The following are the headings of the sections. Part I, the n-dimensional case: (1) critical points, (2) homotopy classes, (3) the principle of stationary points, (4) the category of a closed set with respect to a compact manifold, (5) an estimate of the number of solutions of a variational problem, (6) applications and examples, (7) the category of projective space, (8) applications of homology theory to estimating the category, (9) divisors of a manifold, (10) pseudo-category of pseudo-projective space. Part II, variational problems: (1) deformations of families of curves, (2) the category of families of curves, (3) families of neighborhoods on a sphere, (4) category of families of neighborhoods, (5) proof of some lemmas, (6) theorems on closed geodesics, (7) the operation of contracting curves, (8) the straightening deformation [see below], (9) the preceding deformation for systems of curves, (10) existence proof for an almost geodesic curve in an almost minimal system, (11) application of the theory of category.

The main problem of part II is the proof that at least three different closed geodesics exist on a surface of genus 0 in E^3. If the lengths c_i of the three geodesics are different, then it is still conceivable that, for instance, $c_2=2c_1$ and the geodesic corresponding to c_2 is twice the geodesic corresponding to c_1. The earlier book did not exclude this possibility. The present succeeds in doing so by considering only families of curves without multiple points. This necessitates a proof that the deformations of families of curves can be performed by staying within the realm of simple curves. This proves very complicated and is the content of sections (8) and (9) of part II. There is no reference to work later than the first article either on category or on the calculus of variations in the large, except for one paper of Borsuk in 1936 and one by Èl'sgol'c in 1939. *H. Busemann.*

Charrueau, André. Sur les faisceaux de complexes linéaires. C. R. Acad. Sci. Paris 228, 359–360 (1949).

Les propriétées projectives du faisceau des complexes linéaires [par exemple, la figure polaire d'un complexe du faisceau par rapport à un autre complexe du faisceau, les propriétées des complexes en involution etc.) sont examinées. [Voir aussi G. Koenigs, La géométrie réglée et ses applications, Gauthier-Villars, Paris, 1895, pp. 1–25.]

F. Vyčichlo (Prague).

Charrueau, André. Sur les faisceaux de complexes linéaires. C. R. Acad. Sci. Paris 228, 803–805 (1949).

Charrueau, André. Sur les suites et cycles de complexes linéaires conjugués. C. R. Acad. Sci. Paris 228, 894–896 (1949).

(1) Soient C_1, C_2 deux complexes linéaires non spéciaux du faisceau et soit C_3 le conjugué (à l'aide des polaires réciproques) de C_1 par rapport à C_2. Si μ_1, μ_2, μ_3 sont les valeurs du paramètre λ dans le faisceau correspondant aux C_1, C_2, C_3 et $\lambda_1 \neq \lambda_2$ les valeurs du paramètre λ correspondant aux complexes spéciaux, on a $(\lambda_1\lambda_2\mu_1\mu_2)=(\lambda_1\lambda_2\mu_2\mu_3)$. En posant $(\mu_j-\lambda_1):(\mu_j-\lambda_2)=\alpha_j$, on déduit $\alpha_1\alpha_3=\alpha_2^2$. Pour les deux complexes μ_a, μ_b du faisceau en involution on a $\alpha_a+\alpha_b=0$.

(2) Soit $A+\lambda B=0$ le faisceau des complexes où $A=0$, $B=0$ sont les équations avec les coefficients réels. Soient $\lambda_1 \neq \lambda_2$ les valeurs de λ pour les complexes spéciaux. Partant de C, déterminons dans le faisceau une suite $\mathfrak{S}$ de complexes C_j correspondant à μ_j réels (les valeurs du paramètre λ) et tels que C_j et C_{j+2} soient conjugués par rapport à C_{j+1} quel que soit j. On a $(\lambda_1\lambda_2\mu_j\mu_2)=(\lambda_1\lambda_2\mu_j\mu_{j+1})$ et α_j forment une progression géométrique. L'auteur montre que $\mathfrak{S}$ a cette propriété: Quels que soient j et un autre entier positif m, C_j et C_{j+2m} de $\mathfrak{S}$ sont conjugués par rapport à C_{j+m} ($\mathfrak{S}$ est une suite de complexes linéaires conjugués).

Si la suite $\mathfrak{S}$ contient p complexes distincts et se ferme ($\alpha_{p+1}=\alpha_1$, $\alpha_{p+2}=\alpha_2$, $\cdots$) et λ_1, λ_2 sont réels ou imaginaires conjugués, la suite $\mathfrak{S}$ est un cycle de complexes linéaires conjugués. La condition suffisante et nécessaire pour que $\mathfrak{S}$ soit un cycle est que la raison de la progression géométrique des α_j est une racine primitive $p$ième de 1. Les propriétés géometriques de la suite et du cycle sont aussi mentionnées. Si les coëfficients de C_1 et de C_2 sont réels ou imaginaires, $\lambda_1\lambda_2$ peuvent ne pas être des imaginaires conjuguées. L'auteur définit maintenant un cycle de la manière suivante: La suite $\mathfrak{S}$ constitue un cycle de p complexes distincts quand la raison de la progression géometrique de α_j est égale à une racine primitive $p$ième de 1. L'auteur montre des autres formes simples de cette définition et aussi la signification géométrique dans la géométrie euclidienne. Le cas particulier $\lambda_1=\lambda_2$ est aussi examiné. *F. Vyčichlo* (Prague).

Bompiani, Enrico. Tessuti di curve piane e corrispondenze fra piani. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 7–12 (1949).

This note discusses first some geometrical properties of n-webs in the plane. For example, the web determines a projectivity between the pencil of lines through a point P and the points of intersection of a web-tangent at P with the corresponding web-tangents infinitesimally removed along the given pencil directions. Furthermore a certain $(2n-3)$-web is intrinsically determined by the given one. There is a more elaborate discussion of the case $n=3$ particularly as regards point correspondences between planes. Among other things it is shown that three projectivities can be defined by such a correspondence depending on third order neighborhoods of corresponding points which determine rather simply the six invariants of the third order of the correspondence. Geometric interpretations are an integral part of the method. *J. L. Vanderslice.*

Longo, Carmelo. Trasformazioni puntuali fra due piani proiettivi in una coppia di punti corrispondenti a direzioni inflessionali di specie superiore indeterminate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 68–73 (1949).

In a point-transformation between two projective planes which is regular at a pair of corresponding points (O, O'), there are in general three inflexional directions at O and O' (such that to any curve having an inflexional element in one of those directions at O corresponds a curve with the same behavior at O'). If these directions are indeterminate the given transformation is approximated by a collineation up to and including the neighborhoods of the second order of O and O'. If the inflexional elements of species $s-2$ are considered at O (i.e., the curves whose tangents at O have an s-point contact with the curves), and if the transformation is such that to every $(s-2)$-inflexional element at O corresponds a similar element at O', there are $s+2$ corresponding $(s-1)$-inflexional elements. Such a transformation is approximated by a collineation up to and including the neighborhood of an order $h \geqq 2$, whereas for neighborhoods of order k, where $h \leqq k \leqq s$, it is always possible to approximate the given transformation by a rational transformation of indices $[1, k-1]$.

A further approximation by rational transformations of indices $[1, s]$ is only possible along the $s+2$ inflexional directions of species s (up to and including the neighborhoods of order $s+1$). There are h invariants of order $h+1$; $h-1$, of order $h+2$; $k-1$ invariants of order k such that $h+3 \leqq k \leqq s$; and $2s$ invariants of order $s+1$. The geometric interpretation of these invariants is also given with the help of the preceding rational transformations.

E. Bompiani (Rome).

Dalla Volta, Vittorio. Una questione di geometria riemanniana connessa a un problema di ottica geometrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 64–68 (1949).

The main result may be described as follows. Let $ds^2 \equiv U^{-2}\sum_{\sigma=1}^{n}(dx^\sigma)^2$ be the metric form of a conformally flat Riemann space V_n. A hypersphere in it is given by its center x^σ and geodesic radius x^0. Using a theorem by Fubini the author states that a transformation $\bar{x}^\nu \to x^\nu$ ($\nu=0, \cdots, n$) which maps V_n on a Euclidean space E_n carries hyperspheres into hyperspheres if there is a function V such that $ds'^2 \equiv V^{-2}(ds^2-(dx^0)^2)$ is reducible to $d\bar{s}^2 \equiv \sum_{\sigma=1}^{n}(d\bar{x}^\sigma)^2-(d\bar{x}^0)^2$. This is possible if and only if the curvature tensor of ds'^2 vanishes. Computing its components one sees that U must be of the form which makes the curvature of ds^2 constant.

V. Hlavatý (Bloomington, Ind.).

NUMERICAL AND GRAPHICAL METHODS

***Tables of Scattering Functions for Spherical Particles.** National Bureau of Standards. Appl. Math. Ser., no. 4, xiii+119 pp. (1949).

The purpose of these tables is to facilitate quantitative calculations in scattering phenomena due to spherical particles immersed in a beam of light. The functions i_1 and i_2, defined below, give the angular distribution of intensity and the total light scattered by a small spherical particle as a function of the parameter $\alpha=2\pi r/\lambda$, when the particle radius r is roughly equal to the wavelength λ of the incident light; they are proportional to the intensities of the two incoherent plane polarized components scattered by a par-

ticle illuminated with natural light. The volume consists of four parts. In part I, i_1 and i_2 are tabulated as functions of α [$\alpha=0.5$; 0.6; 1.0; 1.2; 1.5; 1.8; 2.0; 2.4; 2.5; 3.0; 3.6; 4.0; 4.8; 5.0; 6.0], the angle of incidence γ [$\gamma=0°(10°)180°$] and the relative refractive index m of the particle [$m=1.33$; 1.44; 1.55; 2.00], up to four significant figures, with a few exceptions. There is also a table for $\frac{1}{2}\alpha^2K(m;\alpha)$, where K is the total scattering coefficient, defined as the total energy scattered per second per unit cross-sectional area of particle, illuminated at unit intensity [3S]. Definitions: $\beta=m\alpha$; $x=\cos\gamma$; $S_n(x)=(\pi x/2)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x)$; $C_n(x)=(-1)^n(\pi x/2)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(x)$; $\varphi_n(x)=S_n(x)+iC_n(x)$; $\pi_n=\pi_n(x)=dP_n(x)/dx$, where the J's are Bessel functions and $P_n(x)$ is Legendre's polynomial. Further,

$$a_n=(-1)^{n+1}(2n+1)\frac{S_n'(\beta)S_n(\alpha)-mS_n'(\alpha)S_n(\beta)}{S_n'(\beta)\varphi_n(\alpha)-m\varphi_n'(\alpha)S_n(\beta)},$$

$$p_n=(-1)^{n+1}(2n+1)\frac{mS_n(\alpha)S_n'(\beta)-S_n(\beta)S_n'(\alpha)}{m\varphi_n(\alpha)S_n'(\beta)-S_n(\beta)\varphi_n'(\alpha)},$$

$$A_n=a_n/n(n+1);\quad P_n=p_n/n(n+1).$$

Then

$$i_1=|i_1^*|^2=\left|\sum_{n=1}^{\infty}\{A_n\pi_n+P_n[x\pi_n-(1-x^2)\pi_n']\}\right|^2$$
$$=|\Re(i_1^*)+i\Im(i_1^*)|^2,$$

$$i_2=|i_2^*|^2=\left|\sum_{n=1}^{\infty}\{P_n\pi_n+A_n[x\pi_n-(1-x^2)\pi_n']\}\right|^2$$
$$=|\Re(i_2^*)+i\Im(i_2^*)|^2,$$

$$\tfrac{1}{2}\alpha^2K(m;\alpha)=\tfrac{1}{2}\int_0^{\pi}(i_1+i_2)\sin\gamma\,d\gamma$$
$$=\sum_{n=1}^{\infty}\frac{\Re^2(A_n)+\Im^2(A_n)+\Re^2(P_n)+\Im^2(P_n)}{(2n+1)/n^2(n+1)^2},$$

where $\Re$ and $\Im$ denote the real and imaginary parts, and where a prime indicates differentiation with respect to argument. The functions $\Re(i_1^*)$, $\Im(i_1^*)$, $\Re(i_2^*)$ and $\Im(i_2^*)$ are given to four significant figures (some exceptions), while $\Re(A_n)$, $\Im(A_n)$, $\Re(P_n)$, and $\Im(P_n)$ are given to 4S or 6D, for the same values of α, γ, m; $n=1(1)12$.

In part II, the function $K(m;\alpha)$ is tabulated [4S] for $m=1.50$ over a greater range of α values [$\alpha=0.5$; 0.6; 1.0; 1.2; 1.5; 1.8; 2.0; 2.4; 2.5; 3.0; 3.2; 3.4; 3.6; 3.8; 4.0; 4.3; 4.5; 4.8; 5.0; 5.3; 5.5; 6.0; 6.5; 7.0; 7.2; 8.0; 8.4; 9.0; 9.6; 10.0; 18.0; 12.0]. Definitions:

$$K(m;\alpha)=(2/\alpha^2)\Re\left\{\sum_{n=1}^{\infty}(2n+1)(C_n^1+C_n^2)\right\},$$

$$C_n^1=(-1)^{n+1}ia_n/(2n+1);\quad C_n^2=(-1)^n ip_n/(2n+1).$$

Tables for $(2n+1)\Re(C_n^1)$ and $(2n+1)\Re(C_n^2)$ for $n=1(1)16$ [4D].

The tables so far reviewed are concerned with nonabsorbing particles, where m is real. From the tables in part III the coefficient K may be obtained for absorbing spheres of extinction coefficient k varying between 0 and 0.1, and of real refractive index from 1.44 to 1.55. Let

$$F(m;\alpha)=2\alpha^{-2}\sum_{n=1}^{\infty}(2n+1)\{C_n^1(m;\alpha)+C_n^2(m;\alpha)\}$$
$$=K(m;\alpha)+iL(m;\alpha),$$

$$F(m^*;\alpha)=F(m-imk;\alpha)=K(m,-mk;\alpha)+iL(m,-mk;\alpha).$$

Tables for $K(m;\alpha)$ and $L(m;\alpha)$ to 4 or 5S, $m=1.44(0.01)1.55$ with $\alpha=0.5$; 0.6; 1.0; 1.2; 1.5; 1.8; 2.0; 2.4; 2.5; 3.0; 3.2; 3.6; 4.0; 4.5; 4.8; 5.0; 5.5; 6.0; 6.5; 7.0. Polynomial approximations to these functions in the variable $t=m-1.50$. For $K(m,-mk;\alpha)$ and $L(m,-mk;\alpha)$ only polynomial approximations in terms of mk and t.

Finally, part IV contains numerical values of $K(m^*;\alpha)$ and $\bar{K}(m^*;\alpha)$ [3D]; $\Re(C_n^1)$; $\Re(C_n^2)$; $\Im(C_n^1)$; $\Im(C_n^2)$ [4D], for

$m^*=4.21-2.51\,i$; $\alpha=0.10(0.05)1.00(0.1)3.0$;
$m^*=5.55-2.85\,i$; $\alpha=0.10(0.05)1.00(0.1)2.0$;
$m^*=8.18-1.96\,i$; $\alpha=0.100(0.025)1.000$;
$m^*=3.41-1.94\,i$; $\alpha=0.10(0.05)1.00(0.1)5.0$;
$m^*=7.20-2.65\,i$; $\alpha=0.100(0.025)1.000(.05)1.30$;
$m^*=8.90-0.69\,i$; $\alpha=0.10(0.01)0.30(0.005)0.430(0.01)0.60$;

with the definitions as before, and

$$\bar{K}(m^*;\alpha)=2\alpha^{-2}\sum_{n=1}^{\infty}(2n+1)(|C_n^1|^2+|C_n^2|^2).$$

[Reviewer's note: The constant term in the expression for $L(m;7.0)$ on page 81 is misprinted; it should be $-.82600$ instead of $-.28600$; also on page 80, the value of $K(m;6.5)$ obtained from the polynomial for $t=0$ is 12 units of the fourth decimal larger than the tabulated value for $m=1.500$. Similar deviations, though generally smaller, occur at many places in part III.] *C. J. Bouwkamp* (Eindhoven).

Goodwin, E. T., and Staton, J. Table of $J_0(j_{0,n}r)$. Quart. J. Mech. Appl. Math. **1**, 220–224 (1948).

The solution V of Laplace's equation in cylindrical polar coordinates which is to take a constant value V_0 over the surface of the cylinder $r=1$ is, at points inside the cylinder, given by

$$V=V_0+\sum_{n=1}^{\infty}\{A_n\exp(j_{0,n}z)+B_n\exp(-j_{0,n}z)\}J_0(j_{0,n}r),$$

where $j_{0,n}$ is the nth zero of the Bessel function J_0. In this table five-decimal values of $J_0(j_{0,n}r)$ are given for $n=1(1)10$ and the range $r=0.00(0.01)1.00$. These values were calculated to seven decimal places by interpolation from the British Association Mathematical Tables, v. VI, of Bessel functions; they were then differenced in the r-direction on a National accounting machine as a check and all doubtful roundings off were examined; all five decimals printed should, accordingly, be accurate. *S. C. van Veen.*

Fairthorne, R. A., and Miller, J. C. P. Hilbert's double series theorem and principal latent roots of the resulting matrix. Math. Tables and Other Aids to Computation **3**, 399–400 (1949).

The constant in Hilbert's inequality has been improved for finite sums [Frazer, J. London Math. Soc. **21**, 7–9 (1946); Cassels, ibid. **23**, 285–290 (1948); these Rev. **8**, 259; **10**, 434] but the best value is not known. Copsey, Frazer and Sawyer [Nature **161**, 361 (1948); Math. Gaz. **32**, iii–iv (1948); these Rev. **9**, 344; **10**, 367] have reported the results of numerical computations for small indices. The authors report the methods and results of further calculations. *R. P. Boas, Jr.* (Providence, R. I.).

Lahaye, Edmond. Sur la résolution des systèmes d'équations transcendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 809–827 (1948).

The essential feature of this method is that the author generalises the system of equations by introducing a parameter t so that the original system arises for $t=1$. The gen-

eralized system is solved for several values of t: $t_0=0, t_1, \cdots, t_k=1$. The system for t_0 can easily be solved. The system for t_{i+1} is solved by Newton's method starting with the solutions for the system for t_i. *E. Bodewig* (The Hague).

Aprile, Giuseppe. Integrali di Stieltjes, e valutazione planimetrica di espressioni simboliche nel calcolo operatorio. Pont. Acad. Sci. Acta 10, 43–46 (1946).

Nyström has given a method for obtaining the numerical value of a Stieltjes integral of the type (1) $S=\int_a^b g(\theta)dh(\theta)$ by means of a planimeter [Z. Angew. Math. Mech. 14, 276–279 (1934)]. From this the author has derived a method for the numerical evaluation of a symbolical expression of the following type:

$$(2) \qquad W(s)=f(\Delta)\cdot V(s)=\int_0^s V(s-\theta)\cdot G(\theta)\cdot d\theta,$$

where $G(s)$ is the generating function of the operator $f(\Delta)$ ($\Delta\equiv d/ds$). By introducing the generating integral $H(s)$ by means of $G(\theta)=dH(\theta)/d\theta$ the integral (2) is transformed into $W(s)=\int_{s-0}^{s} V(s-\theta)dH(\theta)$, which is a Stieltjes integral of the type (1). *S. C. van Veen* (Delft).

Jaeger, J. C., and Clarke, J. D. A product integraph. J. Sci. Instruments 26, 155–156 (1949).

A brief description of a small machine having some of the features of a differential analyzer, and constructed from parts taken from surplus antiaircraft predictors.

S. H. Caldwell (Cambridge, Mass.).

Kilpatrick, John E., and Kilpatrick, Myra Ferguson. The energy levels and thermodynamic functions of the fourth power oscillator. J. Chem. Phys. 16, 781–787 (1948).

The wave equation for the fourth-power oscillator is transformed into the reduced equation

$$d^2\psi/d\xi^2+(\lambda-\xi^4)\psi=0,$$

$$(a) \qquad \psi(\xi)=Cw(\xi)\sin\left\{\lambda^{\frac{1}{2}}\int_{-\infty}^{\xi} w^{-2}d\xi\right\},$$

where $w(\xi)$ is the solution of the auxiliary differential equation $d^2w/d\xi^2+(\lambda-\xi^4)w-\lambda w^{-3}=0$ (initial conditions $w(0)=1$, $w'(0)=0$). The first six characteristic numbers of (a) have been calculated by a numerical integration method of Milne [Physical Review (2) 35, 863–867 (1930)]. The first, second and third approximations of the Wentzel-Kramers-Brillouin solution of (a) give

$$(b) \qquad n+\tfrac{1}{2}=0.5564178941\lambda^{\frac{1}{2}}-0.04767248525\lambda^{-\frac{1}{2}}+0.01195429069\lambda^{-9/4}.$$

Comparison of the values of λ obtained by numerical integration with those obtained by (b) shows that, for the fourth, fifth and sixth characteristic numbers, the third W.K.B. approximation is correct within 0.0001. In the rest of the paper the molal thermodynamic functions $-(F-H_0)/RT$, $(H-H_0)/RT$, S/R and C/R are calculated as functions of $x=h\nu_0/kT$ over the range $x=0$ to $x=10$.

S. C. van Veen (Delft).

***Lonseth, Arvid T. An extension of an algorithm of Hotelling.** Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 353–357. University of California Press, Berkeley and Los Angeles, 1949. $7.50.

To every element x of a linear vector space L let there belong a "norm" $\|x\|>0$ and let K be a linear operator which leaves L unaltered. Then the author defines the bound of K as $M(K)=\sup \|Kx\|/\|x\|$ for all x of L. Let A be a linear transformation in L and $A=(I-G)A_0$ with $M\equiv M(G)<1$. Then the error of $C_n=C_0\prod_{i=0}^{n-1}(I+G^{2^i})$, $C_0=A_0^{-1}$, is bounded by $M(A^{-1}-C_n)\leqq M(C_0)\cdot M^{2^n}/(1-M)$. This generalised inequality of Hotelling [cf. the same volume, pp. 275–293; these Rev. 10, 574] is applied for several definitions of $\|x\|$ to matrices and to Volterra and Fredholm equations. *E. Bodewig* (The Hague).

Reissner, Eric. Complementary energy procedure for flutter calculations. J. Aeronaut. Sci. 16, 316–317 (1949).

A procedure is outlined for flutter calculations which bears the same relation to the Rayleigh-Ritz procedure as does the theorem of minimum complementary energy in elasticity to the theorem of minimum potential energy. The procedure makes use of the beginning of the Stodola process to determine moment and torque distributions in a wing from arbitrarily assumed modes of deflection and twist. These moments and torques are then used in a variational equation which forms a basis for determining the flutter speed. The method shows promise of giving results of a specified accuracy with a fewer number of modes considered than is necessary with the Rayleigh-Ritz procedure.

S. Levy (Washington, D. C.).

Fettis, Henry E. The calculation of coupled modes of vibration by the Stodola method. J. Aeronaut. Sci. 16, 259–271 (1949).

Spring, Osc. W. Lebensversicherung und Geldwert. Mitt. Verein. Schweiz. Versich.-Math. 49, 96–119 (1949).

The author considers the problem of adjusting insurance policies to variations of the currency value. He makes the following assumptions. (1) The sum insured varies with some index such as the cost of living index. This index is represented by a known function of time. (2) A part of the investment of the insurance company is not affected by the fluctuations of the currency; this part again is given as a known function of time. (3) The premium at any time equals the product of the initial premium and a known function of time. The formulae of actuarial mathematics are then adapted to these idealized conditions. *E. Lukacs.*

MECHANICS

***Signorini, Antonio. Meccanica Razionale con Elementi di Statica Grafica.** Perrella, Rome. Vol. 1, vii+335 pp., 1947, 1200 lire; vol. 2, viii+397 pp., 1948, 1800 lire.

This is a more or less elementary textbook on the subject; the equations of Lagrange do not appear before the last chapter of volume II, this chapter being an introduction to so-called analytical mechanics. The author mentions the influence of Levi-Civita and Maggi. The work is more in accordance with the style of the French classics than with modern American books. In contrast to both and following the Italian tradition more emphasis is laid on vectorial methods. The chapters on the geometry of masses (barycentres, moments of inertia) are rather extensive and the book contains an introduction to graphostatics, but

no problems to be solved by the students. Contents: Volume I, Theory of vectors and introduction to graphostatics; Kinematics of the point; Rigid bodies; Relative motion; Barycentres; Graphical integration; Moments of inertia; Kinematics of systems; Volume II, Fundamental theorems of dynamics; Statics of systems; Mechanics of a point; Stereostatics; Linkages; Motion of threads; Continuous systems; Stereodynamics; Analytical mechanics.

O. Bottema (Delft).

Goldoni, Gino. Sul polo delle accelerazioni nel moto di un corpo rigido libero. Atti Sem. Mat. Fis. Univ. Modena **1**, 12–16 (1947).

Reuschel, A. Über ein einheitliches kinematisches Konstruktionsprinzip zur Ermittlung der Krümmung von Bahnkurven und Hüllbahnen. Ein Beitrag zur Geometrie der Getriebe und zur graphischen Differentialgeometrie ebener Kurven. Österreich. Ing.-Arch. **3**, 9–23 (1949).

This is an exposition of several theorems related to the classical theorem that the motion of a plane (a link or a plate) over a fixed plane may be described as the rolling (without slipping) of a curve in the moving plane over a curve in the fixed plane. This theorem is treated in standard texts on kinematics by Grübler, Beyer and Koenigs. Krames [Maschinenbau, Beilage Getriebetechnik **10**, 481–482 (1942)] gave graphical and linkage constructions for the determination of the centers of curvature of trochoidal curves which are traced by points attached to circles rolling over circles. This construction is extended here to more general curves. The examples given include the conic sections, exponential curves and sine curves.

M. Goldberg (Washington, D. C.).

Artobolevskiĭ, I. I. Mechanisms for enveloping ellipses Doklady Akad. Nauk SSSR (N.S.) **65**, 453–456 (1949). (Russian)

The paper considers the following linkage: A and C are fixed pivots, AB and AD are two rigid bars, B and D are two crossheads, BCD is a right angle. It is shown that the perpendiculars BE (to BC) and DE (to DC) are always tangent to the same ellipse. Three more similar mechanisms are described. A fifth one contains two straight lines enveloping two confocal ellipses. Parabolas and hyperbolas can be similarly enveloped. *A. W. Wundheiler* (Chicago, Ill.).

Levitskiĭ, N. I. Equations of constraints for some two-parameter linkages. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 174–180 (1949). (Russian)

The rigid-bar configurations considered are: (a) $OBMM_1$ with $\angle MBM_1$ constant, O fixed; (b) BMM_1 with $\angle MBM_1$ constant, B sliding along a straight line; (c) MNN_1M_1, with the angles MNN_1, NN_1M_1 right and NN_1 sliding through a fixed point. The coordinates of M_1 are expressed in terms of those of M and appropriate position parameters of the configuration: an exercise in analytic geometry.

A. W. Wundheiler (Chicago, Ill.).

Dimentberg, F. M. An analogy between finite motions of plane and spatial four-bar linkages. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 181–185 (1949). (Russian)

A spatial four-bar linkage L (i.e., one with nonparallel hinges), whose adjacent bars enclose, consecutively, the angles ϕ, θ, χ and ψ, can be matched with a plane four-bar linkage L' so that, if ϕ', θ', χ' and ψ' denote the corresponding angles of L', the relation $\phi=\phi'$ enforces $\chi=\chi'$, $\xi\tan\psi'/2=\tan\psi/2$, $\eta\tan\theta'/2=\tan\theta/2$, where ξ and η are determined by the dimensions of L. The dimensions of L' are explicitly expressed in terms of those of L; only trigonometry (plane and spherical) is involved. The noteworthy fact is, of course, only the existence of these simple formulas, and not the mapping itself. *A. W. Wundheiler.*

Eremeev, N. V. On the premises of the Roberts-Čebyšev theorem. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 115–116 (1949). (Russian)

The Roberts-Čebyšev theorem states a sufficient condition for the generation of the same trajectory by three different four-bar linkages. The author shows that this condition is not a necessary one, and proves that a necessary and sufficient condition is: the point generating the common trajectory and the three instantaneous centers of rotation of the three connecting rods should be collinear at all times. The condition remains valid when some of the pivots are replaced by crossheads. *A. W. Wundheiler.*

Petrokas, L. V. Analytical design of cam mechanisms of automatic machines. Vestnik Inženerov i Tehnikov **1948**, 184–189 (1948). (Russian)

The steps of a semigraphical cam-profile determination are described for three simple special cases (one plane, two spatial) in which the follower is a crank whose motion is prescribed. References are given to the Russian sources of the formulas employed. *A. W. Wundheiler.*

***Rocard, Yves. Théorie des Oscillateurs.** Éditions de la Revue Scientifique, Paris, 1941. viii+223 pp.

The author gives an excellent treatment of the theory of oscillations in general and electrical oscillations in particular. The first chapter treats simple harmonic motion in one dimension, both free and forced with damping. There is also a treatment of finite amplitude oscillations of a pendulum and nonlinear electrical circuits. In the second chapter there is an analysis of the relaxation oscillations including van der Pol's oscillator and also the subharmonic case. The chapter concludes with a detailed treatment of two nonlinear coupled circuits. The third chapter contains an analysis of noninertial (massless) and noninductive systems. In the next chapter the author gives a general treatment of nonholonomic systems. In chapter five the author applies the theory to economic problems. Chapter six contains an analysis of continuous systems (acoustical and electrical) and in chapter seven a short account of regulators and rectifiers is presented. Chapter eight contains a treatment of auto-oscillations of synchronous motors and other servo-mechanical devices. In the last chapter a brief exposition of the vibrations of an airplane wing is presented. *N. Chako.*

***Rocard, Y. Dynamique Générale des Vibrations.** Masson et Cie., Paris, 1943. vii+332 pp.

The author has divided the book into two parts covering respectively the general theory of vibrations and the theory of acoustic waves. The first chapter has a short treatment of simple harmonic vibrations and nonlinear systems with one degree of freedom. In chapter two damping is included. Chapter three contains an analysis of electric and acoustic oscillators and a discussion of the microphone. Chapter four analyses the vibration of bars and membranes. There is a treatment of electrical systems, including vacuum tube circuits, in chapter five. In chapter six the theory of propa-

gation of plane waves in acoustic tubes and in cables is presented in some detail. In chapter seven the author applies Lagrange's equations to a system with many degrees of freedom including the case of forced oscillations and in chapter eight the theory is applied to multiple mechanical and electrical systems. Chapter nine contains a brief treatment of gyroscopic systems. In chapter ten the author presents the theory of the loud speaker, the microphone and the piezo-electric crystal circuit. Chapter eleven contains an account of Heaviside operational calculus as applied to electrical systems and in chapter twelve one finds an account of filters. In chapter thirteen the author discusses the real effects in circuits. The following two chapters contain a detailed study of the stability of self-oscillations in vacuum tube circuits and mechanical circuits under nonlinear conditions. The first part concludes with an account of nonlinear régimes in mechanical and electrical systems with application to automatic regulation, both mechanical and electrical.

The first chapter of the second part contains a discussion of spherical and cylindrical waves based on the wave equation; the results are applied to the problem of radiation from a rigid piston. The chapter concludes with a treatment of reflection and diffraction of waves including reflection from a paraboloidal surface. In chapter two the theory of radiation and diffraction is treated; in particular, the problem of radiation from a pulsating cylinder and a sphere is discussed. Chapter three contains an account of the propagation of plane waves in horns of various types followed by an exposition of their amplification. In chapter five the author discusses the applications of the horn in connection with phonographs and loud speakers. In chapter six the theory of waves of finite amplitude is presented and applied to horn-type loud-speakers. The last chapter briefly discusses electrodynamic microphones. *N. Chako.*

Sanvisens Marfull, Francisco. Contribution to the study of the stationary motion of inextensible cords. Collectanea Math. 1, 63–108 (1948). (Spanish)

The motion of an inextensible cord, with respect to a set of fixed or moving axes of reference, is said to be stationary if the cord always lies on a curve which is fixed with respect to the axes. In the first chapter of this paper the differential equations of motion of a cord are given in several forms, which are then specialized into the forms that apply in the case of stationary motion. The first part of the second chapter deals with stationary motions in the case in which two of the axes rotate about the third. It is found that only a small number of types of stationary motion are possible; and for each of these the dependence of the rotation of the axes upon the time, and properties of the curve upon which the cord lies, are determined. The latter parts of the chapter deal in the same manner with the subcases in which the rotation of the axes is uniform and uniformly accelerated, respectively. The third chapter deals with the case in which the axes of reference have a rectilinear motion, which may be uniform or not. The greater part of this chapter is devoted to discussions of two applications of the general theory. The first of these is concerned with the form of a submarine cable which is being payed out by a ship moving with constant velocity. The second application is concerned with the form assumed by the anchor chain of a ship moored in a stream. The cases in which the anchor chain is uniform and of constant resistance are both considered.

L. A. MacColl (New York, N. Y.).

Krall, G. Dinamica ed aerodinamica dei fili. I. Premesse. Vibrazioni visibili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 11–17 (1947).

In this and the three following notes the author discusses various engineering problems concerning the vibrations of a wire suspended between two points. The particular case which furnishes the motivation for the investigation is that of an electrical transmission line, subjected to disturbances such as wind and a varying ice load. This first note is devoted to a general introductory discussion of the problems, and to an elementary study of the vibrations caused by a force of the form $p(x, t)$, where the function p is sinusoidal with respect to the coordinate x.

L. A. MacColl (New York, N. Y.).

Krall, G. Dinamica ed aerodinamica dei fili. II. Vibrazioni acustiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 17–22 (1947).

[Cf. the preceding review.] When a suspended wire is subjected to a steady wind certain trains of vortices, which were investigated by von Kármán, are generated, and the wire is set into vibration. In this note the author first reviews these familiar physical facts. Then he derives, for engineering purposes, various relations between the constants of the physical system and the parameters which determine the properties of the oscillation.

L. A. MacColl (New York, N. Y.).

Krall, Giulio. Dinamica ed aerodinamica dei fili. III. Problemi non lineari delle vibrazioni visibili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 197–203 (1948).

[Cf. the two preceding reviews.] In this note the author considers the oscillations of a wire stretched between two points in the case in which the amplitudes of the oscillations are large, so that the customary linear theory does not apply. Plausible physical assumptions, together with certain mathematical approximations, lead to the differential equation

$$H_0 w''(1-\tfrac{3}{2}w'^2)-\mu_0\ddot{w}-k_0\dot{w}|\dot{w}|+p(x,t)=0,$$

where H_0, μ_0, and k_0 are constants, $w(x, t)$ is the instantaneous local transverse displacement of the wire, and the primes and dots denote differentiation with respect to the abscissa x and the time, respectively. The term $p(x, t)$, which represents a perturbing force, is taken to be of the form $p_0 \sin(n\pi x/L)\sin \nu t$, where p_0, L, and ν are constants. The author assumes the existence of an approximate solution of the form $w(x, t)=A \sin(n\pi x/L)\sin \nu t$, and determines an appropriate value of the constant A by a least squares process. The results are put into forms suitable for use in engineering work, and their physical implications are discussed in some detail. *L. A. MacColl.*

Krall, G. Dinamica ed aerodinamica dei fili. IV. Problemi non lineari delle vibrazioni acustiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 285–288 (1948).

This is a brief discussion, based upon the contents of the second and third notes reviewed above, of large oscillations of a wire caused by a steady wind. Most of the results relate to the interpretation and numerical estimation of certain parameters which appear in the formulae.

L. A. MacColl (New York, N. Y.).

Hostinský, B. Sur le spectre acoustique de la corde de Lagrange. Publ. Fac. Sci. Univ. Masaryk, no. 297, 19 pp. (1948).

A Lagrangean string is one whose mass is concentrated at a finite number of equidistant points, each point carrying the same concentrated mass. The author investigates the motion of this system, in particular when the motion is caused by repeated "percussions" each of which increases the velocity of each of the concentrated masses by a given amount. An investigation of the distribution of characteristic frequencies shows that there is a largest frequency ν_m, and that the characteristic frequencies become denser as ν increases toward ν_m. The distribution of energy in the frequency spectrum is also investigated, and it is shown that the energy density is proportional to $(\nu_m^2-\nu^2)^{-\frac{1}{2}}$. The energy density for a very large (but finite) number of masses is discussed in two special cases: (i) when a single shock is administered to the first mass of a string in equilibrium, and (ii) when two shocks, giving the same increase of velocity at two different instants to the first mass, are administered. Another problem discussed concerns the velocities which have to be imparted simultaneously to each of the concentrated masses of a string at rest in order to achieve that each characteristic vibration of the string should have the same energy. The paper concludes with some remarks on equipartition of energy in mechanical systems consisting of a large number of identical elements in regular arrangement.

A. Erdélyi (Pasadena, Calif.).

Beránek, J. Sur quelques problèmes se rattachant à la théorie de la corde de Lagrange et de la membrane de Routh. Publ. Fac. Sci. Univ. Masaryk, no. 293, 63 pp. (1947). (Czech. French summary)

In the first part, the author continues Hostinský's work on the Lagrangean string [cf. the preceding review]. He investigates the motion of the system and the distribution of the energy in the spectrum for the motion caused by a simply harmonic force starting at one of its nodes (sinusoidal force) and either continuing indefinitely, or else breaking off at the end of the first half-wave. The motion of the system is also investigated in the case when no forces are acting, but the end of the string is forced to perform the first half-wave of a sinusoidal displacement, and is then fixed again. All forces and displacements are transversal. In the second part some of the theory is carried over to "Routh's membrane," that is, a square-meshed net of weightless elastic strings stretched in a rigid rectangular frame, with equal masses placed at all lattice points.

A. Erdélyi.

Casarini, M. Sul pendolo conico di lunghezza variabile. Boll. Un. Mat. Ital. (3) 3, 251–255 (1948).

It is shown in this note, by means of a particular example which is discussed in detail, that varying the length of a conical pendulum nonadiabatically may result in an initially circular motion's being transformed into an elliptical motion.

L. A. MacColl (New York, N. Y.).

Chaléat, Raymond. Système d'entretien à plusieurs pendules conjugués. C. R. Acad. Sci. Paris 228, 1104–1106 (1949).

Nagabhushanam, K. Configuration-space of a rod sliding inside a rotating ring. Proc. Benares Math. Soc. (N.S.) 8, no. 1, 29–32 (1946).

The author discusses the following problem in the language of geometry of dynamics. A ring rotates about a vertical diameter and a uniform rod is constrained to such a path that its ends remain in contact with the rings. No mechanical interpretation of the quantities discussed is given.

G. F. Carrier (Providence, R. I.).

Peretti, Giuseppe. A proposito di un curioso effetto giroscopico. Atti Sem. Mat. Fis. Univ. Modena 2, 75–86 (1948).

This note gives the theory of a certain toy, consisting essentially of a gyroscope mounted in an egg-shaped case. If the gyroscope is not rotating, the system is in stable equilibrium when the case rests on a horizontal plane with the large end down. However, if the gyroscope is set into rotation, the case turns over, and comes to a stable equilibrium with the small end down. It is shown that this behavior depends upon the presence of friction, and upon the parameters involved satisfying certain inequalities.

L. A. MacColl (New York, N. Y.).

Dobronravov, V. V. On a relation in Euler's problem on the motion of a rigid body about a fixed point. Doklady Akad. Nauk SSSR (N.S.) 65, 143–144 (1949). (Russian)

En rapportant le mouvement aux axes principaux d'inertie et en utilisant les angles d'Euler, l'auteur obtient quatre équations linéaires (les trois intégrales des quantités du mouvement et l'intégrale des forces vives sous forme de Lagrange) à trois inconnues: $d\psi/dt$, $d\theta/dt$ et $d\varphi/dt$. En écrivant la condition de compatibilité, on obtient la relation cherchée, qui représente l'intersection d'une sphère de rayon un avec un ellipsoïde. L'auteur étudie le cas particulier lorsque l'ellipsoïde de giration devient un ellipsoïde de révolution.

M. Kiveliovitch (Paris).

Dobronravov, V. V. Analytic dynamics in anholonomic coordinates. Učenye Zapiski Moskov. Gos. Univ. Mehanika 122, tom II, 77–182 (1948). (Russian)

This dissertation, written like a textbook, is purely formal in content. It presents a step-by-step recasting of the standard formulas, equations and theorems of analytical dynamics into forms valid for first-order anholonomic systems. Tensor symbolism is used throughout. Almost everything in the paper is derived twice: in the first part for holonomic systems and anholonomic coordinates, in the second part for anholonomic systems. The formulas and the proofs are the same in both cases. The transition from the classical to the anholonomic case is a matter of differential-algebraic routine, and a ten-page version of the paper would be quite satisfactory for many readers, especially those accustomed to tensor symbolism. In this form the paper would constitute an almost necessary appendix to every text in analytical dynamics.

The general routine of transition from the classical to the anholonomic case may be briefly described as follows. The symbols $d\pi^\sigma$ and $\partial/\partial\pi^\sigma$ are defined by means of the equations (an index duplicated in a monomial is a summation index) $d\pi^\sigma=A_i^\sigma dq^i$, $\partial/\partial q^i=A_i^\sigma(\partial/\partial\pi^\sigma)$, where q^i are some generalized (true) coordinates of the system. If the first forms are not completely integrable, the symbols π^σ do not denote any numbers, and are said to represent "anholonomic coordinates." Only the defined symbols $d\pi^\sigma$ and $\partial/\partial\pi^\sigma$ enter into the equations, and all first-order relations retain their old form, as if π^σ were true coordinates. Not so with second-order relations. Unless $d\pi^\sigma$ are exact differentials, we have

$$\delta d\pi^\sigma-d\delta\pi^\sigma=\gamma_{\lambda\mu}^\sigma d\pi^\lambda d\pi^\mu,$$
$$\partial^2F/\partial\pi^\lambda\partial\pi^\sigma-\partial^2F/\partial\pi^\sigma\partial\pi^\lambda=\gamma_{\lambda\mu}^\sigma(\partial F/\partial\pi^\sigma),$$

and the γ's disappear whenever π^α are holonomic. The γ's enter linearly in all the second-order equations in the form of an easily predictable additive term. Thus Lagrange's equations assume the form (due to Boltzmann and Hamel)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\pi^\lambda}-\frac{\partial T}{\partial \pi^\lambda}+\gamma^\alpha_{\lambda\nu}\dot\pi^\nu\frac{\partial T}{\partial \dot\pi^\alpha}=Q_\lambda,$$

and Hamilton's,

$$\dot\pi^\alpha=\partial H/\partial p_\alpha,\quad \dot p_\lambda=-\partial H/\partial \pi^\lambda+\gamma^\alpha_{\lambda\nu}p_\alpha(\partial H/\partial p_\nu).$$

Similar modifications occur in all the other formulas.

The most important omission noticed by the reviewer is that of the condition for true anholonomicity (that is, when no transformation of variables will reduce the γ's to zeros). Some interminable computations (in the proof of Poisson's bracket theorem, for instance) can be spectacularly simplified by working with scalar forms. The notations do not make full use of the modern devices in tensor symbolism (due to Schouten and his school) which would result in superior geometric organization of the shape of the formulas.

A. W. Wundheiler (Chicago, Ill.).

Fumi, F. Considerazioni sulla formulazione della equazione di Hamilton-Jacobi. Atti Accad. Ligure **4** (1947), 103–118 (1948).

The usual form of the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t}+H\left(\frac{\partial S}{\partial q}|q|t\right)=0.$$

If we define $V=S+Et$ (E any constant), then V satisfies

$$\frac{\partial V}{\partial t}+H\left(\frac{\partial V}{\partial q}|q|t\right)=E.$$

The author regards this latter form as fundamental, and the central theme of his paper is the existence of a second form

$$\frac{\partial V}{\partial t}+H\left(p\left|-\frac{\partial V}{\partial p}\right|t\right)=E,$$

which he appears to consider new. Actually the equivalent form with $E=0$ was given by Hamilton in 1834, except for the unimportant restriction that H is independent of t [cf. W. R. Hamilton, Mathematical Papers, v. 2, Cambridge University Press, 1940; these Rev. **2**, 23]. *J. L. Synge.*

Aržanyh, I. S. The vortex principle in analytical dynamics. Doklady Akad. Nauk SSSR (N.S.) **65**, 613–616 (1949). (Russian)

This paper presents a summary of an earlier paper by the same author [Trudy Inst. Mat. Meh. Akad. Nauk Uzbek SSR, 1948], which was inaccessible to the reviewer. The close connection between the skew-symmetric bilinear forms $\omega(\delta, d)=\sum_i(\delta p_i dq_i-dp_i\delta q_i)$ and the Hamiltonian systems is a well-known fact. The mechanical interpretation of the latter systems is realized by the motion of a holonomic dynamical system in a conservative field of force. But as the Hamiltonian systems represent only a special case ($r=0$) of more general systems of equations of dynamical systems, which the author calls dynamical systems of the rank $r>0$, it is expected that $\omega(\delta, d)$ will play an important part in the problems of analytical dynamics. The author proposes to transform the d'Alembert-Lagrange principle

$$(1)\qquad \sum_i(\mathbf{F}_i-m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i=0$$

into such a form that $\omega(\delta, d)$ obtains a mechanical interpretation. It is a well-known fact that every modification of the d'Alembert-Lagrange principle uncovers new qualitative properties of the motion of dynamical systems. This fact is very well illustrated by the Gauss principle of least constraint. Also the Hamiltonian principle, being a modified form of the d'Alembert-Lagrange principle, is more useful in the mechanics of continua than its original form (1). The new vortex principle may be expressed as follows. In the motion of a dynamical system in the configuration space at any instant the possible (compatible with the constraints) variation of the associated kinetic energy is to be distributed into two components: the longitudinal (which represents the work done by the exterior forces and the local inertia forces in an arbitrary possible displacement), and the transversal component (the flow of the vortex of impulses per unit of time through an element of area, made by the actual and possible displacements):

$$\delta E=A_\delta(Q+R)+\Omega(d,\delta)/dt,\quad E=-T+\sum_i p_i\dot q_i,$$
$$p_i=\partial T/\partial\dot q_i,\quad \dot q_i=\partial E/\partial p_i,$$
$$\Omega(d,\delta)=\sum_{i,j}\left(\frac{\partial p_i}{\partial q_j}-\frac{\partial p_j}{\partial q_i}\right)(\delta q_j dq_i-dq_j\delta q_i),$$

$$(2)\qquad A_\delta(Q+R)=\sum_i\left(Q_i-\frac{\partial p_i}{\partial t}\right)\delta q_i,$$
$$\delta E=\sum_i\left(\frac{\partial E}{\partial q_i}\delta q_i+\frac{\partial E}{\partial p_i}\delta p_i\right).$$

If one represents the flow of the vortex of impulses p_i and the work of the local forces of inertia $-dp_i/dt$ in the form $\omega(d,\delta)/dt$, then (1) is equivalent to the following equation:

$$(3)\qquad -\delta E+\sum_i Q_i\delta q_i=\omega(d,\delta)/dt.$$

The vortex form (2) of the fundamental principle of dynamics is invariant under contact transformations. As a consequence of the vortex principle (3) the Jacobi-Hamilton method follows, if there exists a force function. If there exists a force function and the system is subject to geometrical constraints, one is led to the method of Suslov [Theoretical Mechanics, Moscow, 1944, p. 461]. In the case when the system is subject to geometrical and kinematical (nonholonomic) constraints the potential method is to be replaced by the vortex method. *E. Leimanis.*

Kasner, Edward, and De Cicco, John. Physical curves in space of n dimensions. Proc. Nat. Acad. Sci. U.S.A. **35**, 201–204 (1949).

A system S_k of ∞^{2n-1} curves in a given positional field of force in Euclidean space of n dimensions consists of curves along which a constrained motion is possible such that the osculating plane at each point contains the force vector F, and the pressure P (along the principal normal to the curve) is proportional (with the constant of proportionality k) to the normal component N of F. In this note the authors state several simple theorems, most of which relate to the curvatures $\kappa_1, \cdots, \kappa_{n-1}$ of the curves of a system S_k. A typical one of these theorems is the following. The rate of variation $d\rho_1/ds$ of the first radius of curvature with respect to arc length is a linear integral function of ρ_1, with coefficients depending upon the point and upon the first two principal unit vectors of the curve. *L. A. MacColl.*

Kasner, Edward, and De Cicco, John. Generalization of Appell's transformation. J. Math. Physics **27**, 262–269 (1949).

In the first part of this paper the authors consider the trajectories of a particle moving in a plane in what they call a generalized field of force, i.e., a field in which the force depends upon the position and direction of motion of the particle. They determine the contact transformations in the plane which, when associated with suitable transformations of the time of the form $dT=dt/F(x, y, p)$, send all families of ∞^3 trajectories into other such families. (In this case the contact transformations turn out to be extended collineations.) They also determine the transformations which send a single family of trajectories into another such family. In the latter parts of the paper they give some similar results relating to various other families of curves, including families of curvature trajectories and families of generalized brachistochrones. *L. A. MacColl.*

Pignedoli, Antonio. Sul problema delle teleferiche. Moto di un corpo rigido pesante, un punto del quale è vincolato a scorrere senza attrito lungo un filo teso, soggetto a piccole oscillazioni intorno alla sua configurazione rettilinea di riposo. Atti Sem. Mat. Fis. Univ. Modena **2**, 149–169 (1948).

Pignedoli, Antonio. Sulle curve naturali di un sistema dinamico. Atti Sem. Mat. Fis. Univ. Modena **2**, 37–59 (1948).

If V_n is a Riemannian space with the line element ds, and if $F(P)$ is a point function defined on V_n, the curves in V_n for which the value of the integral

$$\int_{P_1}^{P_2} F(P)ds$$

is stationary are called natural curves. Various choices of the function $F(P)$ lead to various important natural curves. For instance, V_n may be the configuration space of a conservative holonomic dynamical system, and the natural curves may be the trajectories in this space, corresponding to natural motions of the system with a prescribed value of the energy. Again, the natural curves may be the generalized brachistochrones, i.e., the curves in the configuration space corresponding to constrained motions of the system from one configuration to another for which the energy has a prescribed value, and the time required is a minimum.

In the first part of this paper the familiar generalities concerning natural curves, such as the defining differential equations, are developed by means of vector methods. Particular attention is given to the relations between the theory of natural curves and the Hamilton-Jacobi theory of dynamics, and also to certain equivalences between different problems leading to families of natural curves. In the latter parts of the paper the author gives some new theorems concerning the properties of natural curves. One of these theorems is the following. In the configuration space of a conservative holonomic dynamical system, consider a trajectory and a generalized brachistochrone which correspond to the same value of the energy, and which have the same initial point and the same initial direction. Then at the initial point the geodesic curvatures (and also the geodesic torsions) of the two curves are equal in magnitude and opposite in sign. *L. A. MacColl* (New York, N. Y.).

Pignedoli, Antonio. Sull'esistenza, per un sistema anolonomo, di un integrale lineare nelle velocità lagrangiane. Atti Sem. Mat. Fis. Univ. Modena **1**, 50–58 (1947).

This paper deals with the following problem. Given a material system whose position at any time t can be specified by n Lagrangian coordinates $q_1, \cdots, q_n$ and whose motion is subject to m $(m<n)$ nonholonomic constraints dependent on the time t, under what conditions do the equations of motion of the system admit an integral linear in the Lagrangian velocities $\dot{q}_1, \cdots, \dot{q}_n$? An analogous problem has been treated by Agostinelli [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **80**, 231–239 (1945); these Rev. **9**, 111], when there are ignorable coordinates and the constraints are independent of the time t. By introducing linearly instead of the Lagrangian velocities $\nu=n-m$ parameters ω [caratteristiche cinetiche] and by expressing the equations of motion in terms of quasi-coordinates, the conditions for existence of an integral of momentum with respect to one of the ω's are stated. A special case is discussed in which such an integral in fact exists which, expressed in terms of the Lagrangian velocities, necessitates the existence of an ignorable coordinate. *E. Leimanis* (Vancouver, B. C.).

Pignedoli, Antonio. Sulla applicabilità del metodo di Jacobi della meccanica analitica ai sistemi anolonomi. Atti Sem. Mat. Fis. Univ. Modena **1**, 78–94 (1947).

The principal result of this paper is the following theorem. Consider a material system subject to bilateral constraints without friction and suppose that the position of the system at any instant is specified by n Lagrangian parameters. If in addition the motion of the system is subject to m $(m<n)$ nonholonomic constraints, the solution of the problem of motion can be reduced to the determination of a single function V, considered as a function of the Lagrangian parameters and the time t and containing $n-m$ arbitrary constants, which satisfies a system of $m+1$ partial differential equations of the first order, and to the integration of a system of m ordinary differential equations of the first order in m unknown functions, the latter system being derived from the equations of nonholonomic constraints. Under somewhat less general assumptions the problem has been recently considered by Dobronravov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 477–480 (1939)], and Lampariello [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 734–740 (1942); **4**, 12–19, 20–28 (1943); these Rev. **8**, 234, 235]. *E. Leimanis* (Vancouver, B. C.).

Pignedoli, Antonio. Ancora sulla applicabilità del metodo di Jacobi della meccanica analitica ai sistemi anolonomi. Atti Sem. Mat. Fis. Univ. Modena **2**, 87–95 (1948).

This is a continuation of the paper reviewed above. The author states necessary and sufficient conditions for the consistency of the system of $m+1$ partial differential equations of the first order for the function V and gives the characterization of nonholonomic systems to which the Jacobi-Hamilton method of integration is applicable.
E. Leimanis (Vancouver, B. C.).

Pignedoli, Antonio. Ricerca di soluzioni particolari di un sistema anolonomo in base alla esistenza di integrali o relazioni invarianti. Atti Sem. Mat. Fis. Univ. Modena **1**, 95–118 (1947).

The principal theorem of this paper may be regarded as an extension of a well-known theorem of Levi-Civita [Atti

Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 10_1, 3–9 (1901); cf. also T. Levi-Civita and U. Amaldi, Lezioni di Meccanica Razionale, v. 2, part 2, Zanichelli, Bologna, 1927, chapter 10, § 9; and E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 3d ed., Cambridge University Press, 1927, p. 325], which establishes a connection between the invariant relations or integrals of a holonomic system and certain families of particular solutions of the equations of motion, to nonholonomic dynamical systems. The theorem may be stated as follows. Consider a material system whose position at any time is specified by n Lagrangian parameters and whose motion is subject to m ($m<n$) nonholonomic constraints independent of the time. Suppose that the equations of motion, by introduction of $\nu=n-m$ suitable parameters ω, are written in the canonical form. Then to any set of l ($l<\nu$) invariant relations (or integrals) of the canonical system, which are in involution, if the nonholonomic constraints are such that certain further conditions are satisfied, there corresponds a family of ∞^{m+l} (or ∞^{m+2l}) particular solutions of the canonical system, whose determination depends on the integration of a system of $m+l$ ordinary differential equations of the first order. The theorem is applied to the special case in which the possible motions may be regarded as an extension to nonholonomic systems of those motions which in the case of holonomic systems are known as "motions à la Routh."

E. Leimanis (Vancouver, B. C.).

Agostinelli, Cataldo. Sistemi anolonomi a caratteristiche cinetiche separate e moto di rotolamento di una sfera pesante sopra una superficie generica. Atti Sem. Mat. Fis. Univ. Modena 2, 197–213 (1948).

A statement of conditions under which the equations of motion of a nonholonomic system, by introduction of suitable parameters [caratteristiche cinetiche], break up into two distinct systems of differential equations of the first order, each of which can be solved separately. Two illustrative examples are given: the rolling of a homogeneous and nonhomogeneous sphere on a given surface. In the first case, for example, by choosing as parameters the three components $\omega_1, \omega_2, \omega_3$ of the angular velocity ω of the sphere, referred to the moving trihedral of which the first two axes coincide in direction with the tangents to the lines of curvature $u=$constant and $v=$constant of the surface S at the point of contact M with the sphere and the third axis lies in the direction of the normal N to the surface at the point M, the determination of motion of the sphere is reduced to the integration of a system of five differential equations of the first order, involving the curvilinear coordinates u and v, which define the position of the point M on the surface S, and the three components $\omega_1, \omega_2, \omega_3$ of the angular velocity ω of the sphere. A second system of equations, also of the first order, which connects the components $\omega_1, \omega_2, \omega_3$ with the Eulerian angles and their derivatives, determines as functions of the time t, after the first system is integrated, the three Eulerian angles, which define the orientation of the sphere. In the case of a homogeneous rolling sphere it is shown that the only surfaces for which the component of the angular velocity of the sphere along the normal to the surface at the point of contact is constant are the plane and the sphere.

E. Leimanis (Vancouver, B. C.).

Hydrodynamics, Aerodynamics

Byušgens, S. S. The geometry of a stationary flow of an ideal incompressible fluid. Izvestiya Akad. Nauk SSSR. Ser. Mat. 12, 481–512 (1948). (Russian)

The flow under study is subjected to a conservative force (potential U). The method used is that of the mobile trihedron, analyzed in Cartan's ω-notation of Pfaffians:

$$d\bar{M}=\omega_\alpha{}^\alpha I_\alpha, \quad dI_k=\omega_k{}^\alpha I_\alpha, \qquad \alpha, k=1, 2, 3,$$

where $\bar{M}$ is the radius vector and the I_α are three orthogonal unit vectors. The equation expressing div $\bar{V}=0$ ($\bar{V}$, velocity vector) is:

$$(dV^1-V_r{}^2+V_q{}^3)\omega_0{}^2\omega_0{}^3+(dV^2-V_p{}^3+V_r{}^1)\omega_0{}^3\omega_0{}^1 +(dV^3-V_q{}^1+V_p{}^2)\omega_0{}^1\omega_0{}^2=0 \quad (p=\omega_2{}^3=-\omega_3{}^2, \text{ etc.}).$$

The vortex vector is $\bar{\omega}=\frac{1}{2}$ rot $\bar{V}=\frac{1}{2}\omega^\alpha I_\alpha$.

A number of conditions on these quantities and on the total energy $H=\frac{1}{2}V^2+U+p/\rho$ are geometrically interpreted. Examples are the equations grad $H=2\bar{V}\times\bar{\omega}$, $(V \text{ grad})\bar{\omega}=(\bar{\omega} \text{ grad})\bar{V}$, the family of surfaces of constant total energy and the streamlines which are situated on them, also the case in which the magnitude of the velocity vector is constant. Other cases are that in which $H=f(z)$, hence the surfaces of total energy are parallel planes, and that of so-called minimal ($p_2=q_1$) and rectilinear congruences of stream lines.

The paper ends with a discussion of the spiral flow, that is, a flow for which at every point the direction of the vortex vector coincides with the direction of the velocity ($\bar{\omega}=k\bar{V}$). The type of theorem which is derived in this paper can be gathered from the following example [p. 504]. If the congruence of stream lines consists of straight lines, then it is either a minimal congruence or it is normal to a family of parallel surfaces, and in this second case the vortex lines are orthogonal to the stream lines. *D. J. Struik.*

McVittie, G. C. A systematic treatment of moving axes in hydrodynamics. Proc. Roy. Soc. London. Ser. A. 196, 285–300 (1949).

The author employs the technique of tensor calculus to transform the equations of classical hydrodynamics to moving curvilinear coordinates. Classical equations may of course be viewed geometrically in a space-time in which the fourth coordinate is absolute Newtonian time X^4, but to use tensor techniques an invariant quadratic form is required, and although $(dX^4)^2$ is such a form, it is degenerate. To overcome this difficulty the author uses an artificial kinetic line-element $ds^2=(dX^4)^2-c^{-2}[(dX^1)^2+(dX^2)^2+(dX^3)^2]$, and lets c tend to infinity after using the tensor techniques based on this nondegenerate quadratic form. He discusses in this way the expansion and vorticity of a fluid, the equations of motion and of continuity, and heat transfer. He gives also two examples which refer to the motion of a gas on a spherical rotating earth. *J. L. Synge* (Dublin).

McVittie, G. C. Two-dimensional fluid motion referred to a network of orthogonal curves. Proc. Roy. Soc. London. Ser. A. 196, 301–310 (1949).

The author's method of transforming the equations of hydrodynamics [see the preceding review] is applied to the case of motion parallel to a surface; two coordinates are orthogonal coordinates in this surface and the third coordinate is orthogonal to it. The equations of motion and continuity are worked out in detail and applied to Meyer's equation in aerodynamics [Quart. J. Mech. Appl. Math. 1,

196–219 (1948); these Rev. 10, 338] and to the gradient wind in dynamic meteorology. *J. L. Synge* (Dublin).

Hayes, Wallace D. An alternate proof of the constancy of circulation. Quart. Appl. Math. 7, 235–236 (1949).

Prim, R. C. On the uniqueness of flows with given streamlines. J. Math. Physics 28, 50–53 (1949).

A given flow is here called unique when the only flows with the same streamline pattern have velocity fields which are simply proportional to that of the given flow. For steady flows of an ideal incompressible fluid, in the absence of external force fields, it is proved that: (1) any flow is unique unless it has a constant velocity magnitude along each streamline; (2) all irrotational flows are unique except helicoidal flows obtained by normal superposition of a uniform parallel flow and a potential vortex; (3) all axially symmetric flows are unique except purely axial flows. *D. Gilbarg.*

Ballabh, Ram. Irrotational fluid motions superposable on a motion in circles. Proc. Benares Math. Soc. (N.S.) 8, no. 1, 1–5 (1946).

The author determines the circular motions on which an irrotational motion may be superposed, and determines the superposable irrotational motions. *C. C. Torrance.*

Tandon, H. S. On a heterogeneous fluid motion. Proc. Benares Math. Soc. (N.S.) 8, no. 1, 25–28 (1946).

The author develops a special solution of the equations for superposable fluid motions as formulated by R. Ballabh [Proc. Benares Math. Soc. (N.S.) 3, 1–9 (1941); these Rev. 5, 133]. *C. C. Torrance* (Annapolis, Md.).

*Harrington, R. Paul, and Libby, Paul A. **The shear flow of a perfect fluid about a circular cylinder near a rectilinear boundary.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 37–42. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The author considers two-dimensional flow of a perfect fluid with constant vorticity about a circular cylinder located at a given distance above a straight wall. To solve this problem, the stream-function is separated into two parts: $\psi=\psi_0+\psi_1$, where $\Delta\psi_0=b$ and $\Delta\psi_1=0$, Δ being the Laplacian operator and b the constant vorticity. By introducing dipolar coordinates, the general solution can be expressed in an infinite series involving products of circular and hyperbolic functions. The constants of the series can be determined by the conditions that the straight wall is a stream-line and over the cylinder there is no normal velocity. Numerical results are not included. *Y. H. Kuo.*

Chang, Kwei-Lien. The potential flow around given wing sections. J. Aeronaut. Sci. 16, 306–310 (1949).

The incompressible potential flow about a given obstacle (airfoil) is constructed. The obstacle is transformed into a near circle by a simple mapping and the flow constructed in the "near circle plane" using a Fourier series technique. A numerical example is given. *G. F. Carrier.*

Sedov, L. I. On the theory of the unsteady motion of an airfoil. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1156, 62 pp. (6 plates) (1947).

[Translated from Trudy Central. Aero-Gidrodinam. Inst., no. 229 (1935).] The author investigates the unsteady two-dimensional motion of an incompressible perfect fluid around an airfoil. A system of vortices in the fluid is assumed. The starting point of the investigation is the generalization of the Blasius formulas for nonsteady motion [see also Milne-Thomson, Theoretical Hydrodynamics, Macmillan, London, 1938, section 9, p. 52]. Formulas for the hydrodynamical forces and moment in terms of the complex potential, the circulation, the velocity of the center of gravity of the airfoil, the velocity of the starting point of the line of the velocity discontinuity, are given.

The author determines the complex potential of the motion under the assumption that the function mapping the interior of the unit circle into the exterior of the airfoil is known. He considers in detail two examples: an elliptic wing and a Joukowski profile. He further investigates in detail the properties of the hydrodynamic forces and discusses methods for computing them in the case when (1) throughout the medium the motion is a potential one and the circulation about the wing is equal to 0; (2) the velocity is equal to 0 at infinity and the circulation about the wing is different from 0; (3) a system of point-vortices is present within the fluid. *S. Bergman* (Cambridge, Mass.).

Ertel, Hans. Zur graphischen Konstruktion von Trajektorien in Strömungsfeldern. Z. Angew. Math. Mech. 28, 285–289 (1948). (German. Russian summary)

A new method of successive approximations for the solution of a system of equations to evaluate the small coordinate difference of two neighboring points of a trajectory is stated and transformed into a graphical method for the point by point construction of trajectories. *C. C. Lin.*

Allen, D. N. de G. The formation of closed wakes in fluid motions. Quart. J. Mech. Appl. Math. 2, 64–71 (1949).

This gives the details of a flow bounded in part by cusped free streamlines. The functions describing the flow are quoted in the review of an earlier note by the author on the same subject [Nature 160, 509 (1947); these Rev. 9, 540]. *D. Gilbarg* (Bloomington, Ind.).

Castoldi, Luigi. Sulla generazione di vortici in fluidi perfetti "non omogenei" soggetti a forze di massa conservative. Pont. Acad. Sci. Acta 11, 207–217 (1947).

Teofilato, Pietro. Deduzione dei risultati di una galleria aerodinamica da quelli di un canale idrico. Pont. Acad. Sci. Acta 11, 109–116 (1947).

Moriya, Tomijiro. The theory of lattice composed of airfoils of arbitrary profile form. J. Soc. Appl. Mech. Japan 1, 41–56 (1948). (Japanese. English summary)

Many people believe that the flow through a system of an infinite number of identical airfoils is very similar to the flow through the blades of a propeller or a turbine, if the flow is assumed to be constrained to a cylindrical surface coaxial with the axis of revolution, and that the aerodynamic interference of the blades may be solved by such a latticed airfoil theory. But, in the first place, the distance of two points in the field with latticed airfoils corresponds to the distance along the cylindrical surface in the case of a propeller or a turbine. The influence of a blade does not extend in such a way, but straight through the liquid. In the second place, if the number of blades of a propeller or a turbine is n, the section of a blade appears infinitely many times in every n profiles in the two-dimensional field with

the lattice. Therefore the influence of a blade on any point, in the two-dimensional field, is reckoned infinitely many times along different paths, which, in the actual field, correspond to the paths on the cylindrical surface differing from one another by complete turns of the cylinder. However, the difference of the pressure distribution over the profile from that of a monoplane, and the separating of the streamlines can be shown by the latticed airfoil theory, particularly in the case of a large number of blades. For this purpose, a theory of latticed airfoils of arbitrary profile form is necessary, and is developed here.

From the author's summary.

Toyoda, Kensaburo. Fluid resistance on a flat plate, placed normally to the stream with a pair of vortices in the down stream. J. Soc. Appl. Mech. Japan 1, 142–145 (1948). (Japanese. English summary)

Kondo, Kazuo. Lorentz transformation in aerodynamics. J. Soc. Appl. Mech. Japan 1, 57–63 (1948). (Japanese. English summary)

This paper treats the fundamental problems in the theory of wings and airscrews, for the case when we regard the disturbances due to the lifting system as small. The fundamental equations of the fluid motion may be reduced to the form of the classical wave equation and the velocity of propagation of the disturbances is equal to the velocity of sound. The author's object is to give the aerodynamical interpretation of the characteristic properties of the classical wave equation. In particular, he translates into the language of aerodynamics the well-known properties of Lorentz transformations in the theory of special relativity. For example, the Prandtl-Glauert lift correction corresponds to the Lorentz contraction. The problem of time and causality also has an analogue here. *From the author's summary.*

Kondo, Kazuo. On the fundamental problem in the wing theory of small disturbances. J. Soc. Appl. Mech. Japan 1, 146–153 (1948). (Japanese. English summary)

We obtain, by application of Huygens' principle and the idea of the Lorentz transformation, the fundamental formulas of the acceleration and velocity field. The expression of the former obtained from Kirchhoff's formula concerning the pressure field around the airfoil splits into two parts, one of which is the effect of the camber form of the wing section, while the other is the effect of the thickness. We also obtain the fundamental formula for the oscillating lifting surface and show it to be equivalent to Küssner's.

From the author's summary.

Lukomskaya, M. A. On the flow of a fluid through a hole in an inhomogeneous layer. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 207–208 (1948). (Russian)

The author solves a problem of steady flow of a perfect incompressible fluid subject to specified boundary conditions. Given two (or more) regions D_1 and D_2 occupied by the fluid, it is required to find the characteristic function of flow $W_1(Z)=\varphi_1+i\psi_1$ in the region D_1 and the characteristic function of flow $W_2(Z)=\varphi_2+i\psi_2$ in the region D_2 subject to conditions $\psi_1=\psi_2$, $\varphi_1/C_1=\varphi_2/C_2$, C_1 and C_2 constant. The general case, treated first, assumes for the boundary separating D_1 from D_2 any analytic curve L. From this is deduced a particular case in which L is an ellipse.

A. W. Boldyreff (Albuquerque, N. M.).

Sokolovskiĭ, V. V. On the equations of nonlinear filtration. Doklady Akad. Nauk SSSR (N.S.) 65, 617–620 (1949). (Russian)

Pour étudier la théorie de filtration des eaux, l'auteur pose que la pente hydraulique est une certaine fonction de la vitesse de filtration: $\phi(v)$. Si cette fonction est linéaire en v on retrouve la loi de Darcy qui est valable pour le petit gravier. Pour le gros gravier la fonction n'est plus linéaire. L'auteur établit les équations de filtration non linéaire et montre que pour une fonction ϕ de la forme $vk^{-1}(1-(v/m)^2)^{-\frac{1}{2}}$ ($0\leqq v<m$, k et m ayant les dimensions d'une vitesse) on peut ramener ces équations à une forme très simple.

M. Kiveliovitch (Paris).

Čarnyĭ, I. A. The method of successive displacement of stationary states and its application to the problem of unstationary filtration of fluids and gases. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1949, 323–342 (1949). (Russian)

Florin, V. A. Some of the simplest nonlinear problems arising in the consolidation of wet soil. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1948, 1389–1402 (1948). (Russian)

The author begins with an analysis of a one-dimensional problem of consolidation of a two-phase soil medium, arriving at the result

$$v\frac{\partial \epsilon}{\partial x}+(1+\epsilon)\frac{\partial v}{\partial x}=K\frac{\partial^2 H}{\partial x^2}+\frac{\partial K}{\partial x}\frac{\partial H}{\partial x},$$

where v is the "speed of filtration" of the solid phase, ϵ the ratio of the liquid content to the solid content, K the coefficient of filtration of the liquid, H the hydrostatic pressure. Various changes in form finally lead to what the author considers the most significant form of the equation:

$$\frac{\partial H}{\partial t}+\alpha\left(\frac{\partial H}{\partial x}\right)^2+\beta\frac{\partial H}{\partial x}+\delta\frac{\partial^2 H}{\partial x^2}+\frac{\partial F}{\partial t}=0,$$

where $\alpha=-(1+\epsilon)^2(d/d\epsilon)[K/(1+\epsilon)]$, $\beta=(\gamma_B/\gamma)\alpha+u_0$, γ is the quantity of liquid, γ_B the quantity of solid suspended in the liquid, $\delta=K(1+\epsilon)(\gamma d\epsilon/d\sigma)^{-1}$, σ is the tension in the solid phase, $\partial F/\partial t=-\gamma^{-1}(d/dt)(q+w)+(\gamma_B/\gamma)u_0$, q is the external loading of the upper boundary plane, w the external pressure of the liquid on this plane, u_0 is a constant. Generalizing to the three-dimensional case leads to

$$\frac{\partial H}{\partial t}+\alpha(\text{grad } H)^2+\beta(\text{grad } H, \text{grad } \psi)+\delta\nabla^2 H+\frac{\partial F}{\partial t}=0,$$

where α, β, δ, ψ, and $\partial F/\partial t$ are certain functions of position and time. The balance of the paper deals with an analysis of this equation, and of a number of particular cases corresponding to certain simple sets of boundary conditions.

A. W. Boldyreff (Albuquerque, N. M.).

Kampé de Fériet, J. Sur la décroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine borné ayant pour frontière des parois solides fixes. Ann. Soc. Sci. Bruxelles. Sér. I. 63, 36–45 (1949).

Suppose that an incompressible viscous fluid is contained in an open domain D bounded by a closed surface S, consisting of a finite number of regular surfaces. Assume that (a) the pressure p and its partial derivatives $\partial p/\partial x_i$ are continuous in D, (b) the velocity u_i and partial derivatives $\partial u_i/\partial t$, $\partial u_i/\partial x_j$ are continuous in D, and (c) $\partial^2 u_i/\partial x_j^2$ are

continuous in $D+S$. Then the author establishes the following results. (i) At all time t where the motion is regular, the time rate of change of kinetic energy $E(t)$ satisfies $dE/dt \leqq -6\pi^2\nu L^{-4}E(t)$, where ν denotes the kinematic viscosity and $3L^{-4}=L_1^{-4}+L_2^{-4}+L_3^{-4}$; here L_1, L_2, L_3 are the sides of the smallest parallelepiped containing $D+S$. (ii) In the interval $0\leqq t\leqq T$, where the motion is regular, the kinetic energy satisfies $E(t)\leqq \exp[-6\pi^2\nu L^{-4}(t-t_0)]$, $0\leqq t_0\leqq t\leqq T$. (iii) If the motion is regular for $0\leqq t<\infty$, $\lim_{t\to\infty} E(t)=0$. *Y. H. Kuo* (Ithaca, N. Y.).

Berker, Ratip. Sur l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine spatial borné. C. R. Acad. Sci. Paris **228**, 1327–1329 (1949).

This is an extension of Leray's theorem [J. Math. Pures Appl. (9) **13**, 331–418 (1934)] on the rate of decay of kinetic energy of two-dimensional flow of a viscous incompressible fluid in a bounded domain to the case of three dimensions. By a different method, this has previously been found by Kampé de Fériet [see the preceding review]. *Y. H. Kuo* (Ithaca, N. Y.).

Roscoe, R. The flow of viscous fluids round plane obstacles. Philos. Mag. (7) **40**, 338–351 (1949).

It is shown that the problem of slow viscous flow round a thin plate can be reduced to an electrostatic potential problem, involving the same boundaries. In the case of a plate held perpendicular to the stream of velocity U (electric potential at infinity), the force on the plate is $8\pi\mu CU$, where μ is the viscosity of the fluid and C the electrical capacity of a conductor of the same shape as the plate in the flow. Specific expressions and values of C for circular, elliptic and rectangular plates are given. By the same technique, the flows through an elliptic aperture and a slit are solved and the rate of flow through the aperture is calculated. Finally, the flow produced by a rotating plate is treated. A simple expression for torque in the case of an elliptic plate rotating about its minor axis is derived. The application of the results to the paddle-type viscometer is also described. *Y. H. Kuo* (Ithaca, N. Y.).

Abramowitz, Milton. On backflow of a viscous fluid in a diverging channel. J. Math. Physics **28**, 1–21 (1949).

In the first part of the paper, the author reconsiders Blasius's solution for a viscous flow in a divergent channel. For sufficiently small divergence of the channel, the solution is assumed to be expansible in ascending powers of the slope m of the channel wall. As an initial condition, the flow at the entrance is taken to be Poiseuille flow. The successive approximations have been carried to the second order and a condition for the backflow is found to be $mR=9.24$ against 13.125 given by Blasius, R being the Reynolds number. That the Blasius result was too large has also received experimental confirmation.

Next, the author improves Blasius's method by a "similarity" transformation so that space derivatives in the direction of the flow and normal to it, for large Reynolds number, are not of the same order of magnitude. This consideration leads to a new expansion with respect to a parameter depending both on the slope and the Reynolds number. The problem is then solved with the same zero approximation, i.e., Poiseuille flow at the entrance section. The successive approximations are, however, different. This improved solution enables the author to fix both the location and the slope of the channel necessary for backflow to occur. Numerical results for this critical condition are provided. Blasius's result corresponds to a location infinitely far downstream.

Finally, the behavior of the flow at the entrance to the channel is discussed. For this purpose, only a first approximation of the solution is evaluated approximately near the entrance. This solution becomes singular on the wall and hence is inconvenient for the determination for backflow. Instead, a different expression for the stream-function is found valid at the entrance and near the wall. This solution shows that no backflow is possible at the entrance. *Y. H. Kuo* (Ithaca, N. Y.).

Viguier, Gabriel. Écoulement d'un fluide visqueux incompressible dans un tube mince incliné. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 202–208 (1948). (French. Croatian summary)

Viguier, Gabriel. Répartition tourbillonnaire d'un fluide visqueux incompressible dans le mouvement à deux dimensions. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **3**, 209–212 (1948). (French. Croatian summary)

Wu, Chung-Hua, and Wolfenstein, Lincoln. Application of radial-equilibrium condition to axial-flow compressor and turbine design. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1795, 101 pp. (1949).

The authors are concerned with the flow of a viscous compressible fluid through a multistage axial turbomachine with either straight or tapered walls. Each stage consists of a rotor and a stator, each of which is assumed to have an infinite number of blades which are later replaced by an axisymmetric field of force per unit volume. The dynamic equations and the equation of continuity are set up in vector form and in a scalar form involving the cylindrical coordinates r, θ, z. The authors find it convenient to express these equations in terms of the entropy and the total enthalpy of the gas instead of its pressure and density. This permits an account to be taken of the rate of added external heat and the rate of heat dissipation due to viscosity. In order to make the problem an axisymmetric one, the authors assume the gas properties to be independent of θ. Simplifications result from the neglect of viscous terms when small relative to other terms in the same equation. Further simplifications result from the hypothesis that frictional work remains in the stream and approximately cancels the work done by the axial and radial components of the force. However, the equations continue to involve terms permitting the study of the radial variation of the principal unknowns, the effects of which variation are shown to be negligible (as often supposed by other authors) only when the axial length of each blade row is much larger than its radial length.

The six equations thus obtained involve eight unknowns: the components of the velocity and of the blade force and the entropy and enthalpy. Two additional equations are obtained by selecting two additional hypotheses such as constant work per unit mass over the blade height, constant total enthalpy over the blade height, a free-vortex distribution of tangential velocities or a constant Mach number relative to the blades. Two methods are given for the solution of the equations: (1) finite differences calculated for three stations per stage: before the rotor, between the rotor and stator, and after the stator; (2) a prescribed sinusoidal, meridional component for the flow path of each gas particle. *M. Marden* (Milwaukee, Wis.).

Levič, V. G. The motion of gas bubbles at large Reynolds numbers. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 18–24 (1949). (Russian)

The resistance experienced by a very small gas bubble rising from buoyancy in a viscous liquid is known to be given quite accurately by Stokes's law for the resistance of a solid sphere in slow motion, the surface tension apparently causing the bubble to act like a solid sphere. The author wishes to work out an approximate theory for the case of large bubbles in a viscous fluid. For this purpose he develops a "boundary-layer" theory for the free surface as a perturbation of the corresponding inviscid fluid motion. In order to find the drag of a large bubble, he takes it spherical in form and computes the rate of dissipation of energy, assuming the solution for an inviscid fluid to hold in the viscous case. He obtains double Stokes's formula [this fact is also mentioned by Bateman on p. 157 of his article in the Report of the Committee on Hydrodynamics, National Research Council, Washington, D. C., 1932]. It is stated that preliminary experiments tend to confirm this value for the drag. Although the author discusses the limitations of this formula, it seems to the reviewer that they are much more serious than he supposes. In particular, the assumption of a spherical bubble seems far from the facts for bubble radii at which surface tension can be neglected. The alteration of the bubble shape caused by the combination of the hydrostatic pressure and the dynamic pressure is so extensive that any increase in drag could easily be attributed to increased form drag. There also appear to be some inadequacies in the development of the free surface boundary-layer theory. *J. V. Wehausen* (Falls Church, Va.).

Birkhoff, Garrett. Remarks on streamlines of discontinuity. Revista Ci., Lima **50**, 105–116 (1948).

When fast moving liquid flows past a fixed obstacle (or when the obstacle travels through the fluid) a cavity filled with air or vapour is found to form behind the obstacle, and often a reentrant jet is observed at the downstream part of the cavity. Let p_f be the pressure in the free stream and p_c the pressure in the cavity. Then Prandtl's cavitation number is $K=(p_f-p_c)/\frac{1}{2}\rho V^2$. The author proves that for an axially symmetrical reentrant jet (whether in two or three dimensions) the drag coefficient is $2\{1+K+(1+K)^{\frac{1}{2}}\}\sigma$, where σ is the ratio of the jet cross-section to the obstacle cross-section.

Considering further the penetration of fluid by a fluid jet when a slipstream interface is formed, it is proved that the speeds on either side of the interface are related by $v^2=Av'^2+C$, where A and C are constants. This relation holds not only for incompressible fluids but also for compressible fluids with similar polytropic equations of state, provided the exponent is the same for both fluids.

L. M. Milne-Thomson (Greenwich).

***Courant, R., and Friedrichs, K. O. Supersonic Flow and Shock Waves.** Interscience Publishers, Inc., New York, N. Y., 1948. xvi+464 pp. $7.00.

The first (introductory) chapter begins with a thorough discussion of the basic thermodynamical and mechanical assumptions. From there the differential equations of a flow of a perfect compressible fluid are derived, and the first consequences of these equations (such as the various forms of Bernoulli's law) are discussed. Other media leading to similar mathematical theories (shallow water, solids obeying a nonlinear stress-strain relation) are also noted. The authors strive not to introduce specific assumptions at an unnecessarily early stage. Thus, throughout the whole book, properties depending on the polytropic character of the gas are carefully distinguished from those obtainable from a quite general equation of state. The chapter concludes with the derivation of the equations for flows possessing a smaller number of degrees of freedom than the general unsteady flow in three dimensions, whose treatment presents difficulties beyond the present powers of analysis. For the most part the Eulerian representation is used.

Many of the special flows (those which are either one-dimensional isentropic or spherical isentropic, or steady two-dimensional; irrotational isentropic, or steady rotationally-symmetrical irrotational isentropic) are governed by a system of two quasi-linear partial differential equations of the form (*) $L_i=0$, $i=1, 2$, where L_i is a linear function of the partial derivatives of two unknown functions $u(x, y)$, $v(x, y)$ with coefficients depending on x, y, u, v. These systems are hyperbolic (in the case of steady flows only if the local speeds of the fluid exceed that of sound), and the second chapter develops the theory of such hyperbolic systems. This theory centers around the concept of characteristics and leads to an existence and uniqueness theorem for the Cauchy problem. Other boundary value problems in which the boundary data are given along two curves are also discussed (if these curves are noncharacteristic their time-likeness or space-likeness is of decisive importance), as well as the propagation of discontinuities in the derivatives along characteristics.

If the expressions L_i are homogeneous and the coefficients depend only on the unknown functions (as in the case for one-dimensional and steady two-dimensional flows) equations (*) are reducible to a linear system by interchanging the dependent and independent variables. The authors discuss the singularities of this "hodograph transformation" and stress the importance of the solutions (called simple waves) which are lost during a hodograph transformation since for them the Jacobian $u_xv_y-u_yv_x$ vanishes identically. A fundamental theorem states that the flow adjacent to a region of constant state is a simple wave. The chapter concludes with some remarks on systems with more than two unknown functions.

The following two chapters dealing with one-dimensional and steady two-dimensional flows form the core of the book. The basic model for the one-dimensional flow [chapter III] is the motion of a gas in a tube caused by a moving piston. For continuous flows an integration theory (going back to Riemann) can be given on the basis of the foregoing material. Simple waves appear as rarefaction or compression waves and can be used to build up solutions of certain problems. A basically new situation, however, arises when solutions are found which cannot be continued without singularities, thus indicating that the basic physical assumptions are insufficient for the description of the phenomenon in question. Fortunately it is not necessary to go back to the immensely complicated equations which take into account viscosity and heat transfer. Instead it is possible to introduce as an idealization of the narrow zones in which these effects are appreciable, the so-called shocks, i.e., discontinuities not of the derivatives but of the unknown functions themselves. An unusually complete and penetrating discussion of these is given in section C. In particular, an approximate theory is indicated for weak and medium shocks in which quantities of the order of the third power of the shock strength are disregarded. [For applications of

this method, see the second paper by Friedrichs reviewed below.] Then the authors consider interactions between shocks and simple waves. In some cases an essentially algebraic analysis yields sufficient information, in others (penetration of rarefaction waves), a solution of a boundary value problem of the differential equation is needed. The authors believe that in general the method of finite differences is the most suitable. The chapter concludes with a detailed discussion of detonation and deflagration waves, while wave propagation in an elastic-plastic material is treated in the appendix.

The discussion of a steady supersonic flow in two dimensions [chapter IV] parallels somewhat that of one-dimensional unsteady flows. After a brief discussion of the hodograph method, characteristics (Mach lines in the physical plane) and simple waves (known in this case as Prandtl-Meyer flows) are considered. A description of oblique shocks follows in which the use of shock polars is emphasized. Then interaction and shock reflections are considered A new phenomenon is that of Mach reflection, a configuration consisting of several shock lines and possibly centered simple waves and contact discontinuities (through which no fluid particles pass). For interactions involving simple waves and weak shocks an approximate treatment is proposed which is also applied to a flow around an airfoil. Finally a brief discussion of "facts and conjectures concerning boundary conditions" is given.

The remaining chapters are of a more descriptive character. One deals with the flow in nozzles and jets, the other with three special problems in three-dimensional flow: Ferrari's simplified version of the Kármán-Moore treatment of a rotationally symmetrical flow around a slender projectile, the theory of conical flows (in particular that of the Taylor-Maccoll flow), and unsteady flow with spherical symmetry.

The book is written in the clear and vivid style for which the authors are known. Open problems and unresolved difficulties are carefully noted, and the reader is never left in doubt as to whether he is presented with a mathematical theorem or with a conjecture based on physical experience. Numerous excellent drawings, a bibliography of almost 200 titles and a carefully prepared index add to the value of this treatise which is certainly destined to become standard.

L. Bers (Princeton, N. J.).

Friedrichs, K. O. Formation and decay of shock waves. Communications on Appl. Math. **1**, 211–245 (1948).

This paper presents an approximate treatment of various problems involving shocks of small or medium strength in unsteady one-dimensional and steady two-dimensional gas flows. This method consists in expanding all relevant quantities in power series in the shock strength and in neglecting terms of order higher than the second. In particular, the change of entropy across the shock front is ignored, so that in two dimensions the flow behind the shock is considered irrotational. The basic ideas of this approach were developed elsewhere [cf. the book by Courant and the author reviewed above, sections 72–76, 137–139].

The one-dimensional problems considered deal with a gas in a cylindrical tube closed at one end by a piston. If the piston is pushed into the gas a shock will develop. If the piston is arrested or retracted, the resulting rarefaction wave will catch up with the shock and lead to shock decay. If a piston is first pushed into the gas, then retracted and arrested when the original position is reached, then there will appear a wave zone bordered by two shock fronts ("N-wave") [T. W. M. DuMond, E. R. Cohen, W. K. H. Panofsky and E. Deeds, J. Acoust. Soc. Amer. **18**, 97–118 (1946)]. In all these cases the author gives approximate formulas for the position and strength of the shock as functions of time and for the width of the wave zones, and for the pressure distribution. For instance, if the piston first is pushed into the gas with constant velocity, and then stopped at the time t_R, the width of the shock wave (distance between the shock front and the tail of the rarefaction wave) increases like $(t-t_R)^{\frac{1}{2}}$ while the shock strength decreases like $(t-t_R)^{-\frac{1}{2}}$.

In two dimensions the author considers the shocks emanating from the trailing and leading edges of an airfoil. He gives approximate formulas for the shape and strength of these shocks at large distances from the axis of the airfoil. In particular, the shock strength is shown to decrease as the reciprocal of the square root of the distance.

L. Bers (Princeton, N. J.).

Friedrichs, K. O. On the non-occurrence of a limiting line in transonic flow. Communications on Appl. Math. **1**, 287–301 (1948).

The differential equations of a two-dimensional potential gas flow become linear when the velocity components u and v are considered as independent variables. A solution of these equations represents a physically possible flow only if the Jacobian $J=x_uy_v-x_vy_u$ does not vanish (x and y are the Cartesian coordinates in the plane of the flow). If $J=0$ on a line L in the hodograph plane ((u, v)-plane), the image of L in the physical (x, y)-plane is called a limiting line. A limiting line can not occur in the subsonic region. The opinion has been expressed that the breakdown of a potential transsonic flow past an airfoil or in a convergent-divergent channel is due to the appearance of limiting lines in the mathematical representation of the flow. The author shows that this is not so. He considers a family of flows of the desired type which depend continuously upon a parameter t. The boundaries in the physical plane (which may depend upon t) are assumed to have a uniformly bounded curvature. The functions describing the flow, say $x(u, v)$ and $y(u, v)$, are assumed to be nonconstant analytic functions for all values of t considered (but no analytic dependence on t is assumed). It is further assumed that no cavitation takes place for $t\leqq t_0$, and that $J\geqq 0$ for $t<t_0$, and $J>0$ in the sonic and transonic regions. The assertion is that the same inequalities are true for $t=t_0$. The author first shows that J cannot vanish on the boundary as a result of the assumptions made about the curvature. He then shows that the hypothesis that $J=0$ at some point in the flow together with the assumption that $J\geqq 0$ in the neighborhood of the point permits one to deduce that all derivatives of the functions describing the flow vanish at the point considered. In the case when the Mach number at this point is exactly 1 use is made of an algebraic lemma (asserting that a certain polynomial in two real variables assumes negative values). The proof of this lemma is supplied in an appendix by D. A. Flanders. [The fact that a limiting line cannot appear within the supersonic region or along a wall was proved also by Nikolsky and Taganov [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 481–502 (1946); Brown Univ., Grad. Div. Appl. Math., Translation **A9-T-17** (1948); these Rev. **8**, 237; **9**, 631; for another translation cf. the

fourth following review], who did not, however, discuss the possibility of its appearance on the sonic line.]

L. Bers (Princeton, N. J.).

Lax, Anneli. Decaying shocks. A comparison of an approximate analytic solution with a finite difference method. Communications on Appl. Math. 1, 247–257 (1948).

The problem considered is that of a bore moving with constant speed into still water. At the time $t=0$ a vertical wall is inserted in the water causing a depression wave which catches up with the bore and causes its decay. This is the water wave analogue of the gas dynamical problem of shock decay (a bore corresponds to a shock, a depression to a rarefaction) in which, however, no entropy changes are to be considered. The problem is treated first by the approximate method described by Friedrichs [cf. the second preceding review], then by the finite difference method. It turns out that the results of the two methods are in good agreement for $h_1/h_0=1.2$, 1.5, 1.8 and even 3.0 (h_1 is the height of water behind the "shock," h_0 that of the still water).

L. Bers (Princeton, N. J.).

***Tollmien, Walter. The direct hodograph method in the theory of the flow of compressible fluids.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 89–110. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

If a steady potential gas flow is given in the "physical" (x, y)-plane, x and y considered as unknown functions in the hodograph plane satisfy linear partial differential equations. This method of linearizing the gas-dynamical equations the author calls the "direct hodograph method." [It is not essentially different from the linearization by means of the Legendre transformation.] In this paper only supersonic flows are considered and characteristic coordinates are used in the hodograph plane. Using the direct hodograph method the author discusses (1) limiting lines in general; (2) limiting lines in transonic flow [in this connection Friedrich's result [cf. the second preceding review] on the nonoccurrence of limiting lines should be mentioned]; (3) the propagation of discontinuities along Mach lines; (4) small disturbances in the (transonic) purely circulatory flow past a circle [cf. Taylor, J. London Math. Soc. 5, 224–240 (1930)].

L. Bers (Princeton, N. J.).

Keller, Joseph B. On the solution of the Boltzmann equation for rarefied gases. Communications on Appl. Math. 1, 275–285 (1948).

This paper is concerned with the deduction of macroscopic gas dynamical equations by integrating the Maxwell-Boltzmann equation of the kinetic theory of gases. As is customary in the kinetic theory, only the simplest possible case is considered (essentially a homogeneous, inert, monatomic, perfect gas). Most previous deductions of this type depend upon an expansion of the distribution function in a series of powers of λ/a, where λ is a properly defined "mean free path" and a is a typical macroscopic dimension. Thus, the results if valid at all will be appropriate to a rather dense gas which is still sufficiently rare that the basic assumption that only binary encounters occur, from which the Maxwell-Boltzmann equation is deduced, is acceptable. The series obtained by these methods cannot be expected to converge, since as $\lambda\to 0$ the gaseous state is relinquished. Generalizing a procedure of G. Jaffé [Ann. Physik (5) 6, 195–252 (1930)], the author adopts the more reasonable course of expanding the distribution function in powers of a/λ. Convergence is at least possible, since as $\lambda\to\infty$ there results a motion without encounters ("free molecular flow"), in no way inconsistent with the hypotheses of the theory. The results calculated from this theory are thus appropriate to very rare gases, the approximation improving in proportion to the degree of rarefaction. The author indicates that the coefficients in this series may be uniquely determined and that if the series is assumed convergent then it is possible to determine a unique distribution function which takes on prescribed values on a given 6-dimensional surface $\phi(x, y, z, u, v, w)=0$ in the 7-dimensional space of the independent variables, provided that (1) the same value is prescribed at all points of intersection of $\phi=0$ with a given particle trajectory; (2) each trajectory intersects any solid or liquid boundary surface but once; (3) upon such boundary surfaces, a rather general boundary condition (permitting specular reflection, diffuse reflection, part specular and part diffuse reflection, absorption, condensation, etc.) later proposed by the author is satisfied. The author indicates a method of calculating the force and moment excited by such a rarefied gas upon a body; while no details are given, he states that some results calculated in special cases by elementary physical arguments are substantiated at first approximations by his theory, but that supposedly more refined conjectures of this type yield results which do not appear to be equivalent to his. He calculates first approximations to the velocity and density as functions of position and time for expansion into a vacuum, and compares the results with the corresponding exact simple wave solution of the ordinary gas dynamical equations given by Courant and Friedrichs [cf. the fifth preceding review]. The agreement is good on the whole; the discontinuities in slope which appear in the gas dynamical solution do not appear in the kinetic theory solution.

C. Truesdell.

Nikolskii, A. A., and Taganov, G. I. Gas motion in a local supersonic region and conditions of potential-flow breakdown. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1213, 35 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 481–502 (1946); these Rev. 8, 237. For another translation cf. Brown Univ., Grad. Div. Appl. Math., Translation A-9-T17 (1948); these Rev. 9, 631.

Wecken, Franz. Stosswellenverzweigung bei Reflexion. Z. Angew. Math. Mech. 28, 338–341 (1948). (German. Russian summary)

When a shock wave falls obliquely on a wall, the primary and reflected waves may form a V with vertex on the wall and moving along it, or a Y configuration may result, abruptly or gradually, with the center of the Y moving forward and outward. The former case is always observed when the angle α between primary shock and wall is small, the latter (called Mach reflection, as opposed to "regular" reflection) when α is large. Cranz and Schardin suggested that regular or Mach reflection would take place according as the stream deflection determined by shock strength and incidence angle α was less than or greater than the maximum allowed by the shock polar diagram. On the other hand, Schultz-Grunow advanced the criterion that the reflection would remain regular so long and only so long as the shock propagation velocity determined by the pressures in front of the primary and behind the reflected wave remained less than the velocity of the vertex along the wall. The author

resolves this ambiguity by showing that the Cranz-Schardin criterion is correct for weak shocks, the Schultz-Grunow criterion for strong shocks. In the latter case the transition is continuous, while in the former it is abrupt.

D. P. Ling (Murray Hill, N. J.).

Richter, Hans. Die Stabilität des Verdichtungsstosses in einer konkaven Ecke. Z. Angew. Math. Mech. **28**, 341–345 (1948). (German. Russian summary)

The shock attached to the point of a wedge symmetrically placed in a supersonic stream has two mathematically possible configurations: the "strong" solution of high intensity and the "weak" solution of lesser intensity. In practice only the weak solution is observed, and numerous explanations have been advanced to account for this preference. The author dismisses these and advances an argument to show that no grounds for distinction exist in the realm of stationary flows; that, in short, the question is one of stability under disturbance.

The author explains the stability of the weak solution and the instability of the strong on the basis of the shock polar diagram. His argument is roughly as follows. For the weak solution, an increase in shock strength entails a sharper turning of the streamlines through the shock, for the strong shock a less sharp deflection. When the attached shock is weak, a sudden increase of pressure behind it tends to separate the streamlines from the wall, an expansion wave is set up, and the pressure drops. A sudden drop in pressure, on the other hand, drives the streamlines into the wall, which tends to restore the pressure. For the strong configuration the reverse is true: pressure increases tend to build up until the shock detaches itself from the nose, pressure decreases to continue until the stable weak configuration is attained. If for some reason only the strong stationary solution exists, this line of reasoning suggests a continuing oscillation of the shock between two nonstationary extremes, one attached, the other detached. If a narrow wedge is followed by a rectangle much broader than the base of the wedge this situation exists, and the resulting high frequency oscillations have been observed by Oswatitsch.

D. P. Ling (Murray Hill, N. J.).

Sokolov, A. A. On the solution of the non-linear hydromechanical equation by the perturbation method. Vestnik Moskov. Univ. **1947**, no. 4, 77–82 (1947). (Russian. English summary)

The author treats the equation describing the (adiabatic) motion of a gas in a tube with one fixed end and with a freely moving piston of mass M. The differential equation reads: $\xi_{tt}=c^2(\xi_x)^{-\gamma-1}\xi_{xx}$. The initial conditions are: $\xi_t=0$, $\xi_x=$constant for $t=0$, $0\leqq x\leqq a$. The boundary conditions are: $\xi=0$ for $x=0$, $\xi_{xx}=(\mu/a)\xi_x$ for $x=a$. Here ξ is the amplitude, x and t Lagrangian variables, c the acoustic speed for the (constant) initial state, γ the ratio of specific heats, a the initial distance from the piston to the fixed end, $\mu=m/(\gamma M)$, m being the mass of the gas [cf. A. E. F. Love and F. B. Pidduck, Philos. Trans. Roy. Soc. London. Ser. A. **222**, 168–226 (1922)]. Assuming an expansion in powers of μ and neglecting μ^3 and terms of higher order, the author obtains an approximate solution of the form $\xi=\xi_1(t)\xi_2(x)+u$. Here $\xi_2=x-(\mu/6)(x-x^3/a^2)+\cdots$, ξ_1 is determined by the relation

$$\int_1^t \xi_1^{(\gamma-1)/2}[\xi_1^{\gamma-1}-1]^{-\frac{1}{2}}d\xi_1=(2vk)^{\frac{1}{2}}t,$$

where $v=1/(\gamma-1)$, $k=c^2a^{-2}\mu[1-\frac{1}{4}(1+v)\mu/(2v)]$. For small values of t, $\xi_1\approx 1+kt^2/2$; for large values of t, $\xi_1\approx(2vk)^{\frac{1}{2}}t$. The formulas for the additional term u are too complicated to be reproduced here. This term represents a standing wave which disappears as $t\to\infty$.

L. Bers.

* **Poritsky, H. Polygonal approximation method in the hodograph plane.** J. Appl. Mech. **16**, 123–133 (1949) = Proc. Symposia Appl. Math., Vol. I, pp. 94–116. American Mathematical Society, New York, N. Y., 1949. $5.25.

As first shown by Chaplygin, the linear equations of flow in the hodograph plane can be reduced to Laplace's equation by a change in the scale of speed of flow, if the true isentropic pressure-volume curve is approximated by a straight line. The author extends and improves this method by approximating the isentropic curve by straight line segments. The hodograph plane is then divided into concentric regions; in each region, there is an appropriate velocity scale for reducing the equation to Laplace's equation. The physical conditions that the coordinates of the space points and the fluid velocity at these points which correspond to boundary points between regions in the hodograph plane must be continuous, give a condition on the gradients of velocity potential and stream function with respect to the transformed velocity at the boundary points. Using these results, the author first shows that for simple particular solutions his approximation is very close in the magnitude of stream function as compared with the exact solution. A method for images for constructing solutions is then given. Finally graphical results for vortex and source in the hodograph plane are shown.

H. S. Tsien (Pasadena, Calif.).

Poritsky, H. Linearized compressible flow. Quart. Appl. Math. **6**, 389–403 (1949).

The homogeneous solutions of Laplace's equation already discussed by the author [Quart. Appl. Math. **6**, 379–388 (1949); these Rev. **10**, 534] are applied to the study of linearized supersonic compressible flow. By a transformation due to Chaplygin, the perturbation velocity components of a conical flow are represented as the real parts of analytic functions of a complex variable. As an example, the lift decrease near the tip of a trapezoidal wing in a supersonic stream is obtained. The extension of the method to nonconical supersonic flows is also indicated.

E. N. Nilson.

Craggs, J. W. The breakdown of the hodograph transformation for irrotational compressible fluid flow in two dimensions. Proc. Cambridge Philos. Soc. **44**, 360–379 (1948).

This paper investigates the nature of the singularities arising from the hodograph transformation in transonic flows. The author shows that in the subsonic region there are only isolated singularities. On the other hand, in the supersonic region two different types of line singularities occur: namely, the "branch line" and the limit line. The "branch line" is (1) a Mach line on which the curves of constant speed and constant direction of flow touch the Mach lines of the other family; (2) on the "branch line" the acceleration of the flow is finite and (3) in the hodograph plane the stream lines touch the same epicycloid. The limit line is characterized by the fact that (1) it is an envelope of Mach lines, (2) curves of constant speed and constant direction of flow are enveloped by the limit line; (3) the acceleration is infinite and (4) in the hodograph plane the stream lines touch different epicycloids. Since the flow

in the neighborhood of a "branch-line" exists, the appearance of a limit line is taken as a criterion for the breakdown of potential flow. *Y. H. Kuo* (Ithaca, N. Y.).

Goldstein, S., Lighthill, M. J., and Craggs, J. W. On the hodograph transformation for high-speed flow. I. A flow without circulation. Quart. J. Mech. Appl. Math. **1**, 344–357 (1948).

In a previous paper [Proc. Roy. Soc. London. Ser. A. **191**, 323–341 (1947); these Rev. **9**, 391] Lighthill indicated a method of constructing (in the hodograph plane) steady two-dimensional flows past closed profiles which reduce to given incompressible flows for vanishing free stream Mach number [cf. also T. M. Cherry, ibid. **192**, 45–79 (1947); these Rev. **9**, 554]. An alternative method is presented in this paper for the case of a circulation-free flow "corresponding" to an incompressible flow past a circle. The authors start with the remark that for this incompressible flow the complex potential as a (multiple-valued) function of the conjugate complex velocity ζ is represented by the four series

$$w_1=\sum_0^\infty \frac{(n-1)(n-\frac{1}{2})!}{(-\frac{1}{2})!n!}\zeta^n, \quad w_2=\sum_0^\infty \frac{(n+\frac{1}{2})(n-\frac{1}{2})!}{(-\frac{1}{2})!n!}\zeta^{1-n},$$

$$w_3=-w_1, \quad w_4=-w_2,$$

($x!=\Gamma(x+1)$) in different parts of the (two-sheeted) hodograph; and that the analytic continuation of w_1 leading to w_2, w_3, w_4 may be obtained by considering the integral

$$\frac{1}{2\pi i}\int_{-\infty i}^{+\infty i}\frac{(v-1)(v-\frac{1}{2})!(-v-1)!}{(-\frac{1}{2})!}(-\zeta)^v dv.$$

For a compressible fluid they begin with a solution of Chaplygin's well-known equation of the form $\psi=\Im W_1$,

$$W_1=\sum_0^\infty \frac{(n-1)(n-\frac{1}{2})!}{(-\frac{1}{2})!n!}\psi_n(\tau)e^{-n(s_1+i\theta)}.$$

Here τ, θ are Chaplygin's hodograph variables,

$$\psi_n(\tau)=\tau^{n/2}F_n(\tau)$$

Chaplygin's solutions and s_1 a parameter depending on the stream Mach number. The analytic continuation of W_1 is achieved by means of the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(v-1)(v-\frac{1}{2})!(-v-1)!}{(-\frac{1}{2})!}(-1)^v\psi_v(\tau)e^{-v(s_1+i\theta)}dv.$$

The final solution has the four representations W_1+L, W_2, $-W_1-L$ and $-W_2$, where

$$W_2=i\sum_0^\infty \frac{(n+\frac{1}{2})(n-\frac{1}{2})!}{(-\frac{1}{2})!n!}\psi_{(\frac{1}{2})-n}(\tau)e^{(n-\frac{1}{2})(s_1+i\theta)},$$

$$L=\sum_2^\infty \frac{(-1)^{n+1}(n+1)!(-n-\frac{3}{2})!}{(-\frac{1}{2})!}C_n\psi_n(\tau)e^{n(s_1+i\theta)};$$

the C_n are numerical constants. The series for W_2 converges for $\tau>\tau_1$ (τ_1 the free stream speed), the series for L everywhere. Possible generalizations of this method, and connections with Lighthill's general solution, are discussed.

L. Bers (Princeton, N. J.).

Lighthill, M. J. On the hodograph transformation for high-speed flow. II. A flow with circulation. Quart. J. Mech. Appl. Math. **1**, 442–450 (1948).

This is a continuation of a previous paper by the author, Goldstein, and Craggs [see the preceding review]. This time it is assumed that the basic incompressible flow past a circle has nonvanishing circulation. If the development of part I is followed, one obtains a stream-function which is multiple-valued in the physical plane. A modification of the method, however, overcomes this difficulty. The author concludes that the present method is inferior to the one developed by him previously [Proc. Roy. Soc. London. Ser. A. **191**, 323–341, 341–351, 352–369 (1947); these Rev. **9**, 391, 350, 391]. *L. Bers* (Princeton, N. J.).

Lighthill, M. J. The flow behind a stationary shock. Philos. Mag. (7) **40**, 214–220 (1949).

A uniform supersonic flow is disturbed slightly to produce a steady flow which, while adiabatic (and inviscid) need not be isentropic or isoenergetic. The linearized equations are developed in terms of the small disturbance velocities (u, v, w). These and the adiabatic assumption result in a new vector (u', v, w), where u' is proportional to the pressure defect, which is shown under many general conditions of interest to be irrotational. Moreover, its associated potential satisfies the same linear equation as holds for the true velocity potential in linearized isentropic flow. A knowledge of the entropy and energy of a streamline establishes the relation between u' and u along that streamline. In particular, this situation prevails for the flow behind a shock ahead of which the flow is uniform, provided the tangential velocity at the shock is along a principal direction of curvature.

The latter condition is certainly fulfilled in the two-dimensional case, and the author takes a closer look at this. The flow behind the shock created by a symmetric wedge is uniform so long as the sides of the wedge are straight. Any curvature of the side creates a disturbance which is propagated along a characteristic, meets the shock, and is reflected back along a characteristic of the other set. Using the (u', v, w) potential and the shock equations, the author derives the reflection coefficient for the upwash. A calculation shows that for incident flows in the ordinary supersonic regime, the reflectivity is small, but it may be close to unity for hypersonic flows. The same analysis shows that to the first order shock and wedge have in a sense similar shapes.

A final section shows that if the leading edge of the wedge is blunted, there will be a concentration of entropy, an entropy boundary layer, near the wedge, which has certain properties in common with the viscous boundary layer.

D. P. Ling (Murray Hill, N. J.).

Linnell, Richard D. Two-dimensional airfoils in hypersonic flows. J. Aeronaut. Sci. **16**, 22–30 (1949).

Certain simple supersonic flows are investigated from the point of view of Tsien's similarity law of hypersonic flow [J. Math. Physics **25**, 247–251 (1946); these Rev. **8**, 237]; namely, those associated with a straight oblique shock, a Prandtl-Meyer expansion, and the flow past aerofoils in the form of a flat plate, a double wedge, a parabolic profile or an isosceles triangle. Pressure, lift, drag, moment coefficients in supersonic flow at large Mach numbers are plotted as functions of the Tsien similarity parameter K and of the angle of attack, as well as in other ways. The flow between the front and rear shocks is treated as isentropic in the calculation of surface pressures. It seems to the reviewer that this renders the results worthless at least for the parabolic profile, and that some investigation is needed into whether for the other profiles the reflections from the front

shock of Mach waves from the surface intersect it again: for the importance of this point in hypersonic flow see Lighthill [preceding review]. *M. J. Lighthill.*

Kuo, Y. H. On the hodograph method. J. Aeronaut. Sci. 16, 189 (1949).

This note contains the author's answer to the criticism by Lighthill [Proc. Roy. Soc. London. Ser. A. 191, 352–369 (1947); these Rev. 9, 391] and Cherry [ibid. 192, 45–79 (1947); these Rev. 9, 544] of Tsien's and his paper [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1063 (1947); these Rev. 8, 237]. *L. Bers* (Princeton, N. J.).

***v. Mises, R., and Schiffer, M. On Bergman's integration method in two-dimensional compressible fluid flow.** Advances in Applied Mechanics, edited by Richard von Mises and Theodore von Kármán, pp. 249–285. Academic Press, Inc., New York, N. Y., 1948. $6.80.

Chaplygin's equation for the stream-function of a subsonic gas flow in the hodograph plane may be written in the form (1) $\overset{*}{\psi}_{\theta\theta}+\overset{*}{\psi}_{\lambda\lambda}+f(\lambda)\overset{*}{\psi}=0$. Here θ is the inclination of the velocity vector, λ a certain given function of speed ($-\infty<\lambda<0$, $\lambda=-\infty$ corresponding to the Mach number $M=0$, $\lambda=0$ to $M=1$); $\overset{*}{\psi}=\psi(1-M^2)/\rho$, where ψ is the stream-function and ρ the density; $f(\lambda)$ is a given function determined by the pressure-density relation. The authors present a method of generating solutions of (1) based on Bergman's theory of integral operations. Their presentation is direct and self-contained. A solution of (1) is given by a series (2) $\overset{*}{\psi}=\sum g_n(\theta,\lambda)G_n(\lambda)$, where g_n are harmonic functions, g_0 arbitrary, (3) $2(\partial g_n/\partial\lambda)=-g_{n-1}$, and the G_n are determined by the equations (4) $G_0=1$, $G'_{n+1}=G''_n+fG'_n$. The recursion equations (3) can be solved by setting $g_n=\Re\phi_n(\zeta)$, $\zeta=\lambda+i\theta$, where ϕ_0 is an arbitrary analytic function and $(n-1)!2^n\phi_n(\zeta)=\int_0^\zeta\phi_0(\zeta)(\zeta-t)^{n-1}dt$. The domain of convergence of (2) is determined. The authors also discuss the behavior of the G_n at $\lambda=0$, and various ways of extending the representation (2) over a wider domain.

In the second part of the paper, the authors set (5) $f(\lambda)=C\lambda^{-2}$ ($C>0$). In this case, (2) may be written in the form

$$(6)\quad \overset{*}{\psi}(\lambda,\theta)=g_0(\lambda,\theta)-\Re\int_0^\zeta\phi_0(t)\frac{d}{dt}\{H[\alpha\cdot\zeta\cdot\gamma;\,(2\lambda)^{-1}(\zeta-t)]\}dt,$$

where H is a hypergeometric function, $\alpha=\frac{1}{2}-(\frac{1}{4}-C)^{\frac{1}{2}}$, $\beta=\frac{1}{2}+(\frac{1}{4}-C)^{\frac{1}{2}}$, $\gamma=1$. If $\phi_0(0)=0$, this simplifies to

$$\overset{*}{\psi}(\lambda,\theta)=\Re\int_0^\zeta\phi_0'(t)H[\alpha\cdot\beta\cdot\gamma;\,(2\lambda)^{-1}(\zeta-t)]dt.$$

The domain (in the subsonic region) in which (6) represents a solution is limited only by the singularities of ϕ_0. Then the authors discuss the (approximate) pressure-density, density-speed, and Mach number-density relations relations resulting from assumption (5). It turns out that the speed will never vanish. *L. Bers* (Princeton, N. J.).

Sauer, Robert. Beziehungen zwischen der Theorie der Flächenverbiegung und der Gasdynamik. Arch. Math. 1, 263–269 (1949).

The author points out a differential-geometrical interpretation of the theory of steady two-dimensional and unsteady one-dimensional gas flows. Let $\mathfrak{s}$ be the position vector of a surface. In an infinitesimal deformation, each surface element experiences an infinitesimal rotation given by the vector $\epsilon\mathfrak{x}$, and, in general, $\mathfrak{x}$ may be interpreted as the position vector of another surface. For a two-dimensional flow, let u and v be the velocity components, p the pressure (so that $p=f(u^2+v^2)$, f being a given function). If $\mathfrak{x}=(x,y,\psi)$ is the vector describing an infinitesimal deformation of the surface $\mathfrak{s}=\mathfrak{s}(-v,u,p)$, then $\psi=\psi(x,y)$ satisfies the equation for the stream-function. In the case of a one-dimensional isentropic flow, the same relationship exists between the vectors $\mathfrak{s}=(-q,u,p)$, $\mathfrak{x}=(x,t,\psi)$. Here u is the speed, p the pressure, x the space coordinate, t the time, $q=u^2/2+\int\rho^{-1}d\rho$ ($\rho=$density), $q=g(q+u^2/2)$, g being a given function, and ψ a "stream-function" defined by the equations $pu=-\psi_t$, $\rho=\psi_x$. *L. Bers* (Princeton, N. J.).

Kawata, Sanji. On a method of solution of supersonic flows. J. Soc. Appl. Mech. Japan 1, 2–7 (1948). (Japanese)

Germain, Paul, et Bader, Roger. Étude de certains mouvements vibratoires harmoniques à l'aide d'une correspondance avec les mouvements homogènes. C. R. Acad. Sci. Paris 228, 1201–1202 (1949).

By writing the disturbance velocity potential of a supersonic flow in the general direction x and Mach number M, with circular frequency ω, as $\varphi\cdot\exp[i\omega(t-xM^2/V\beta)]$, the differential equation for φ is the damped wave equation

$$(A)\qquad \frac{\partial^2\varphi}{\partial x^2}-\frac{1}{\beta^2}\left(\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}\right)+k^2\varphi=0,$$

where $\beta^2=M^2-1$, $k^2=\omega^2\sin^2\alpha/V^2\cos^4\alpha$ and α is the Mach angle. The author sketches two methods of solution for the problem of the oscillating thin wing. The first method is based on the fundamental solution and in the general case with "subsonic" wing boundary can be usually carried out by the inversion formula of Krasil'ščikova [Doklady Akad. Nauk SSSR (N.S.) 58, 761–762 (1947); these Rev. 10, 77]. The second method is that of separation of variables. The solutions of (A) are $\varphi_n=x^nH_n(\chi,\theta)F_{n+\frac{1}{2}}(2k^2u)$, with $x=\beta r\chi$, $y=r\cos\theta$, $z=r\sin\theta$, $2u=x^2-\beta^2r^2$; $F_{n+\frac{1}{2}}(t^2)$ is proportional to $J_{n+\frac{1}{2}}(t)/t^{n+\frac{1}{2}}$, $F_{n+\frac{1}{2}}(0)=1$ and $\bar{\varphi}_n=x^nH_n(\chi,\theta)$ are the corresponding solutions for the steady flow ($k=0$).

H. S. Tsien (Pasadena, Calif.).

Reutter, F. Eine ebene Potentialströmung in der Umgebung der Schallgeschwindigkeit. Ing.-Arch. 16, 299–306 (1948).

The author assumes a nearly uniform flow and a relation between the velocity and the density which leads to the potential equation (1) $-\phi_v\phi_{zz}+\phi_{vv}=0$ which is of mixed type. A Legendre transformation yields the linear equation (2) $-U\varphi_{vv}+\varphi_{UU}=0$. Particular solutions of (2), especially those expressible by Bessel functions, are discussed and various possibilities of applying these to the computation of flows are described. No definitive procedure is given, however, and the author promises numerical results at a later date. [Equation (1) was used by von Kármán [J. Aeronaut. Sci. 14, 373–402 (1947), p. 396; these Rev. 9, 111] to derive a similarity law for transsonic flows. The author seems unaware of the existing literature on the Darboux-Tricomi equation (2) and its applications to gasdynamics.] *L. Bers* (Princeton, N. J.).

Moeckel, W. E. Use of characteristic surfaces for unsymmetrical supersonic flow problems. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1849, 49 pp. (1949).

Beginning with the equation of three-dimensional potential flow of a perfect gas, in general orthogonal coordinates, the author finds the equations of the characteristic surfaces, which are real for supersonic flow, and the equations of velocity differentials along characteristic surfaces. He then presents an approximate calculation in which differentials are replaced by differences and the characteristic surfaces, locally, by their tangent planes. This leads to a standard procedure for progressing from one source curve to another, which can be employed to construct a three-dimensional network of characteristics in practical supersonic-flow problems. The calculation is, of course, relatively tedious. It is assumed for simplicity that the flow is everywhere supersonic and irrotational and that initial conditions are known along a surface that separates a uniform flow field from the disturbed one under consideration. The problem of introducing shock waves into the flow is not treated, again to simplify the work, and it is proposed that these be replaced by isentropic compressions. The appropriate initial conditions for typical engineering problems are discussed briefly. Finally the theory is applied to the small-perturbation type of flow, for which the equations may be linearized. This results in great simplifications, of course, and is proposed for step-by-step difference calculations of small-perturbation flow around boundaries of general shape. In appendices, the special case of asymmetrical conical flow is treated. The method of the paper actually breaks down here but can be salvaged if initial conditions near but off the body can be found or improvised. Detailed calculations of an elliptic cone are presented. *W. R. Sears* (Ithaca, N. Y.).

Heaslet, Max. A., Lomax, Harvard, and Spreiter, John R. Linearized compressible-flow theory for sonic flight speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1824, i+45 pp. (1949).

The equations of isentropic potential flow, linearized under the assumption that the disturbance velocities due to a body are small compared to the speed of the body, yield nontrivial solutions consistent with this assumption in some cases of sonic flight speed. Several such cases are treated here, the first being unsteady plane flow. The problem of sudden change of incidence at constant speed is considered, both for subsonic and supersonic flight speed, using the analogy with steady three-dimensional supersonic flow. The "indicial lift" function is used to calculate the lift in an oscillatory case. The second case considered is the steady three-dimensional one, first for Mach numbers near unity and then for Mach number equal to unity. This part of the paper includes a discussion of wave drag and vortex drag in general. It is found that solutions consistent with the small-disturbance approximation can be found at sonic flight speed for lifting wings without thickness and for some shapes with thickness, such as infinitely long sweptback wings. The differential equation for the perturbation potential ϕ becomes $\phi_{yy}+\phi_{zz}=0$, where y, z are the coordinates normal to the stream direction. The load distributions on two thin wings, triangular and sweptback, at sonic speed, are shown.

The authors are careful to claim only that their solutions for sonic speed are consistent with the theory. There remain some questions of their physical validity. It is clear from physical arguments that some steady sonic flows are impossible, such as about two-dimensional cylinders. It is also clear that the methods of this paper are capable of handling certain other sonic cases, such as the infinite swept cylinder; it seems reasonable to the reviewer that there are classes of problems for which this is true. *W. R. Sears.*

Pfeiffer, F., and Meyer-König, W. The characteristics method applied to stationary two-dimensional and rotationally symmetrical gas flows. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1211, 20 pp. (1949).

[Translated from Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB), Forschungsber. no. 1581 (1942).] The paper gives the characteristic lattice-point integration of the equations for steady two-dimensional and axially symmetrical supersonic flows of compressible fluid. The method is developed for a perfect gas but otherwise very general: the flow can have rotation and the "total energies" along the streamlines need not be equal. The last section considers flow with compression shock. [Curiously, the translator uses the word "laminar" for "irrotational" and "turbulent" for "rotational."] *H. S. Tsien.*

Broderick, J. B. Supersonic flow round pointed bodies of revolution. Quart. J. Mech. Appl. Math. 2, 98–120 (1949).

This is a calculation of axisymmetric flow about bodies of revolution, carried to higher orders of approximation than has been done previously. The flow is assumed to be isentropic and irrotational within the approximation used here; this is not correct, but the approximation seems to be supported by the results of the paper reviewed below. The velocity potential is calculated in the form

$$\phi = t^2\phi_2+t^3\phi_3+t^4\phi_4+t^5\phi_5+t^6\phi_6+\cdots$$
$$+t^4\log t\,\phi_4'+t^6\log t\,\phi_6'+\cdots \tag{1}$$
$$+t^6\log^2 t\,\phi_6''+\cdots,$$

where t denotes the maximum thickness ratio of the body and the primes do not denote differentiation. This form is substituted into the complete equation of motion, and a series of linear equations for the ϕ_i, ϕ_i', ϕ_i'' is obtained. The logarithmic terms in (1) are needed in order to satisfy the boundary condition at the surface of the body. This condition is satisfied up to, but not including, $O(t^6\log^2 t)$. The pressure on the body is calculated neglecting terms of the same order.

To assess the accuracy of the solution, it is applied to the case of a circular cone. Numerical results are presented graphically for three cone semi-vertex angles: 5°, 10°, and 15°, in the form of surface pressure coefficient plotted against stream Mach number M. The exact solution [e.g., Taylor and Maccoll, Proc. Roy. Soc. London. Ser. A. 139, 278–297, 298–311 (1933); or Tables of Supersonic Flow Around Cones, Massachusetts Institute of Technology, Department of Electrical Engineering, Center of Analysis, Tech. Rep. no. 1, 1947; these Rev. 8, 540] is plotted for comparison in each case, as is the "first approximation," which includes t^2 and $t^2\log t$ terms, but neglects $O(t^4\log^2 t)$. These graphs show that the new formula gives a substantial improvement; in fact it is nearly indistinguishable from the exact result for the 5° cone. For the 10° cone it is quite good up to $M=3$; for the 15° cone it is useful only for $M<2$.

W. R. Sears (Ithaca, N. Y.).

Broderick, J. B. Supersonic flow past a semi-infinite cone. Quart. J. Mech. Appl. Math. **2**, 121–128 (1949).

In the paper reviewed above, the author employed an approximate method of solution that seems incapable of giving any information about the shock wave at the nose. In fact, the solutions differ from undisturbed flow only inside the nose Mach cone and the series diverges on that Mach cone. In the present paper, the symmetrical flow past a cone is considered. For the region near the body the solution is expanded in a form similar to that given above, where now t is the tangent of the cone semi-vertex angle, and a series of linear equations for the successive potential functions results. The boundary condition at the cone can be satisfied, but once more the Mach cone represents a singularity. For the region near the shock wave another solution is used, consisting of a power-series expansion in the angle measured from the shock cone. This solution is joined to the asymptotic form of the first solution in the region just inside the Mach cone. The potential for points near the body is finally obtained in an expression correct to $O(t^6 \log^2 \frac{1}{2}y\alpha)$, where y is the customary conical coordinate r/x (r and x being cylindrical coordinates) and α denotes $(M^2-1)^{\frac{1}{2}}$. The corresponding formula for the pressure coefficient on the cone surface is the same as that found in the paper reviewed above. The author infers that this would be true in general for pointed bodies of revolution that have convex surfaces. The shock-wave angle deduced from this calculation, as well as the entropy rise and shock strength, agree with results of Lighthill [same Quart. **1**, 309–318 (1948); these Rev. **10**, 413]. *W. R. Sears* (Ithaca, N. Y.).

Ward, G. N. Supersonic flow past slender pointed bodies. Quart. J. Mech. Appl. Math. **2**, 75–97 (1949).

This paper gives the theory of supersonic flow past slender bodies of general cross-section. The linearized potential equation is solved in terms of Bessel functions by the operational technique. An expansion for small r, where r is the radial coordinate in a cylindrical system, yields a harmonic expression for the potential in an (r, θ)-plane. At this point the author makes an investigation of accuracy, and finds that this expression gives surface velocities within a factor $1+O(t^4 \log t)$, where t denotes the maximum thickness ratio. There are conditions on the shape of the body, including the requirement that the radius of curvature of any cross-section be $O(1/d)$ where the section is convex outward, d being the maximum diameter of the section. There is no restriction where the section is concave.

The drag is calculated from the rate of change of momentum through a cylinder of small radius enclosing the body. The result, to $O(t^6 \log^2 t)$, is expressed by certain integrals involving the function $S(s)$, i.e., the cross-sectional area distribution along the axial (streamwise) coordinate, and contributions from the base. When the body is pointed at both ends, or is of cylindrical form near its tail and this part has zero incidence, the formula is identical with von Kármán's formula for the drag of pointed symmetrical bodies of revolution [Accad. Italia. Atti 5. Conv. Sci. (Volta Congress), 222–283 (1936)]. The lateral force and moment are also computed, and special attention is given to the cases of bodies of revolution and bodies of elliptic section, with flat bases, at incidence. In the appropriate special cases, the results of previous investigators are verified. The theory is also applicable to flat wings of very small aspect ratio; here the results of R. T. Jones [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1032** (1946); these Rev. **9**, 114] are obtained. Finally the author considers winged bodies of revolution, and obtains results that include formulas for the wing-body interference effect in a rather general case. *W. R. Sears* (Ithaca, N. Y.).

Frankl, F. I. Effect of the acceleration of elongated bodies of revolution upon the resistance in a compressible gas. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1230**, 8 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 521–524 (1946); these Rev. **8**, 238. For another translation cf. Brown Univ., Grad. Div. Appl. Math., Translation **A9-T-19** (1948); these Rev. **9**, 631.

Prim, R. C. On the existence of steady gas flow in plane isothermal streamline patterns. Bull. Amer. Math. Soc. **54**, 1009–1012 (1948).

Let a two-dimensional steady flow of a perfect gas possess streamlines which could also be the streamlines of an incompressible potential flow. Then these streamlines are either radial straight lines, or parallel straight lines, or concentric circles. [Under the hypothesis that the gas flow is irrotational the same result has been obtained by D. Gilbarg, J. Math. Physics **26**, 137–142 (1947); these Rev. **9**, 252.]

L. Bers (Princeton, N. J.).

Chapman, Dean R. Laminar mixing of a compressible fluid. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1800**, 19 pp. (1949).

The author carries out a theoretical investigation of the velocity profiles for laminar mixing of a stream of compressible fluid with a region of fluid at rest, assuming that the Prandtl number is unity. The power law of temperature variation is assumed. For the 0.76 power, the calculation is made by an iteration process similar to that used by von Kármán and Tsien. For the first power, the case of a compressible fluid may be obtained from the incompressible case by a transformation of variables [see L. Howarth, Proc. Roy. Soc. London. Ser. A. **194**, 16–42 (1948); these Rev. **10**, 270]. *C. C. Lin* (Cambridge, Mass.).

Piercy, N. A. V., Whitehead, L. G., and Tyler, R. A. The laminar boundary layer. New series and approximate solutions of the equations for steady two-dimensional flow. Aircraft Engrg. **20**, 354–359 (1948).

The first part of the paper gives a power series solution of the boundary layer in two-dimensional flow. This type of series expansion has been used before, but the independent variables along and perpendicular to the solid boundary are chosen differently. The method is restricted to the boundary layers of cylinders which have a rounded nose. For symmetrical flow it is found to be applicable to appreciably greater ranges than the earlier series solutions. The approximate method uses two integral relations obtained by integration of the boundary layer equation across the layer once (von Kármán) and twice, respectively. The velocity distribution contains two parameters and has a form suggested by the series solution. It does not, however, satisfy the boundary condition for the fourth derivative at the solid surface. Good agreement with Hartree's elaborate solutions are found, except in the immediate vicinity of the break away. *C. C. Lin* (Cambridge, Mass.).

Whitehead, L. G. An integral relationship for boundary layer flow. A new relation for use either simultaneously with, or in replacement of, the von Kármán momentum integral with satisfactory results. Aircraft Engrg. **21**, 14–16 (1949).

The two integral relations mentioned in the preceding review are examined by using a polynomial of the fourth degree. It is found that the relation obtained by two integrations gives results comparable to the original Kármán-Pohlhausen method. The use of these two relations jointly gives better results for retarded flow. *C. C. Lin.*

Kalikhman, L. E. Heat transmission in the boundary layer. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1229**, 43 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 449–474 (1946); these Rev. **8**, 418.

Rubesin, M. W., and Johnson, H. A. A critical review of skin-friction and heat-transfer solutions of the laminar boundary layer of a flat plate. Trans. A.S.M.E. **71**, 383–388 (1949).

Shibuya, Iwao. Theory of the laminar boundary layer for sudden change of flow and its applications. I. J. Soc. Appl. Mech. Japan **1**, 94–100 (1948). (Japanese. English summary)

When an immersed solid body, or conversely, the main stream, is accelerated from uniform velocity U_0 to $U_0+U_1(t)$, a vortex sheet appears, since the fluid in contact with the solid body must have the same velocity as the boundary. The vorticity in the sheet diffuses into the fluid from the boundary, and the boundary layer grows. Assuming U_1 to be small compared with U_0, we can linearize the differential equation of the boundary layer and solve it by operational calculus. The problem is treated on the assumption that a solution of this equation consists of two parts, the outer solution and the inner solution. For the former, we can neglect the influence of the reduction factor u_0/U_0, as its correctness is required in the outer parts of the boundary. For the latter, however, since its correctness is required near the wall, we assume for the first approximation $u_0=v_0=0$. *From the author's summary.*

★**Sears, W. R. The boundary layer of yawed cylinders.** Proc. Symposia Appl. Math., Vol. I, pp. 117–118. American Mathematical Society, New York, N. Y., 1949. $5.25.

Cf. J. Aeronaut. Sci. **15**, 49–52 (1948); these Rev. **9**, 476.

Görtler, H. Zur Approximation stationärer laminarer Grenzschichtströmungen mit Hilfe der abgebrochenen Blasiusschen Reihe. Arch. Math. **1**, 235–240 (1948).

The author discusses the accuracy in the method of solving laminar boundary layer equations by expansion in power series of x (distance along the solid boundary). If the velocity $U(x)$ in the free stream is expressed by

$$(1) \qquad u(x)=\sum_{k=1}^{\infty} u_k x^k, \qquad u_1\neq 0,$$

and the stream function is represented by

$$(2) \qquad \psi(x,y)=\sum_{s=1}^{\infty} F_s(y)x^s$$

(y being distance perpendicular to the solid boundary), it is known that $F_s(y)$ can be expressed in terms of the u_k's and a certain set of universal functions. By terminating series (1) at k terms and series (2) at m terms, the boundary condition

$$(3) \qquad \nu(\partial^3\psi/\partial y^3)_{y=0}=-U(x)U'(x)$$

(ν being the kinematic viscosity) is in general not satisfied. The author points out that this condition can be exactly satisfied if $m\geqq 2k-1$. Since the next boundary condition $(\partial^4\psi/\partial y^4)_{y=0}=0$ is automatically satisfied, the author suggests that the solution (2) with m terms is satisfactory, if the condition (3) is satisfied exactly or to a sufficient degree of approximation. In the case of the circular cylinder, Hiemenz's measurement shows that $U(x)$ may be accurately represented with $k=5$. By using Ulrich's calculation [not yet published] of the functions necessary for $F_5(y)$, the author shows that the velocity distribution in the boundary layer can be accurately represented up to the point of separation by the present method. *C. C. Lin.*

★**Lees, Lester. Stability of the laminar boundary layer in a compressible fluid.** Proc. Symposia Appl. Math., Vol. I, pp. 74–80. American Mathematical Society, New York, N. Y., 1949. $5.25.

A fuller account appeared in Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1360** (1947); these Rev. **9**, 116.

Taylor, G. I., and Batchelor, G. K. The effect of wire gauze on small disturbances in a uniform stream. Quart. J. Mech. Appl. Math. **2**, 1–29 (1949).

The authors give a comprehensive account of the effect of woven wire gauze on small disturbances in a uniform stream. The relevant aerodynamic properties of the gauze are characterized by two coefficients k and α connected by the relation $\alpha=1.1(1+k)^{-\frac{1}{2}}$ obtained from a more general one of Dryden and Schubauer. The coefficient k is the ratio of the pressure drop across the gauze divided by the dynamic pressure for a stream normal to the gauze, and the coefficient α is the ratio of the angle of the incoming (slightly oblique) stream to that of the outgoing stream, both measured relative to the normal to the gauze. The analysis is carried out for a uniform stream normal to the gauze, which produces small disturbances both upstream and downstream. Linearization of the equations is introduced.

For steady disturbances it is found that the reduction factors for longitudinal and transverse velocity disturbance are respectively $(1+\alpha-\alpha k)/(1+\alpha+k)$ and α. The first formula is found to agree with experiments of Collar and MacPhail when the relation $\alpha=1.1(1+k)^{-\frac{1}{2}}$ is introduced. The longitudinal disturbance is wiped out for $k=2.8$.

The effect of the gauze on turbulence is found in a similar manner, and is found to depend on the spectrum of turbulence. Calculations for a usual spectrum give results numerically very close to those derived from a consideration of independent longitudinal and transverse steady disturbances as given above. These are compared with experiments and with the reduction factor $(1+k)^{-1}$ proposed by Dryden and Schubauer for all three components of turbulence energy. The experimental difference in the longitudinal and transverse reduction factors is not conclusive enough to discriminate between the two theories. This fact is attributed to the effect of wind-tunnel contraction and the tendency toward isotropy of the turbulent motion, besides the effect of the gauze. *C. C. Lin* (Cambridge, Mass.).

Agostini, Léon. La fonction spectrale de la turbulence isotrope. C. R. Acad. Sci. Paris **228**, 736–738 (1949).

The author considers the spectral function $\int_0^\infty k^{-1}F(k)dk$, where k is the wave number and $F(k)$ is Heisenberg's spectrum. Usual results are discussed in terms of this function. *C. C. Lin* (Cambridge, Mass.).

Agostini, Léon. Sur quelques propriétés de la fonction de corrélation totale. C. R. Acad. Sci. Paris **228**, 810–811 (1949).

The author considers the correlation function obtained by a contraction of the Kármán-Howarth correlation tensor. Usual results are discussed somewhat more simply in terms of this function. *C. C. Lin* (Cambridge, Mass.).

Chandrasekhar, S. On the decay of isotropic turbulence. Physical Rev. (2) **75**, 1454–1455 (1949).

The spectrum of turbulence has been obtained following Heisenberg's assumption for the transfer of energy. It is stated that the solution leads to a cut-off of the spectrum at high frequencies instead of the inverse seventh power law stated by Heisenberg. [See the following review.] *C. C. Lin* (Cambridge, Mass.).

Chandrasekhar, S. On the decay of isotropic turbulence. Physical Rev. (2) **76**, 158 (1949).

This note shows that the author's analysis of the spectrum of turbulence agrees with Heisenberg's result for large frequencies. This clarifies an apparent contradiction announced in the note reviewed above. *C. C. Lin.*

Hardtwig, E. Die hydrodynamischen Gleichungen der ausgeglichenen Bewegung in turbulenten Strömungen. Z. Angew. Math. Mech. **28**, 290–296 (1948). (German. Russian summary)

The fundamental equations for the mean motion in a turbulent flow are derived on the basis of four assumptions. The final equations appear to be of the same form as those for laminar motions; the viscosity coefficient is replaced by a much larger exchange coefficient. *C. C. Lin.*

Kampé de Fériet, J. Sur l'écoulement d'un fluide visqueux incompressible entre deux plaques parallèles indéfinies. Houille Blanche **3**, 509–517 (1948).

The turbulent motion between infinite parallel plates is discussed systematically, with hypotheses explicitly stated. In particular, it is shown that if the velocity distribution satisfies the laws $U/U_\tau=\phi(U_\tau y_0/\nu)$ and $(U_c-U_m)/U_\tau=$ constant, the logarithmic laws of von Kármán hold. In these formulae, U_τ is the friction velocity, U is the mean velocity distribution with maximum value U_c at the middle of the channel, U_m is the average of U across the channel, y_0 is the distance from the wall, and ν is the kinematic viscosity coefficient. *C. C. Lin* (Cambridge, Mass.).

Hess, Robert V. Study of unsteady flow disturbances of large and small amplitudes moving through supersonic or subsonic steady flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1878**, 99 pp. (1949).

Stoker, James J., Jr. The breaking of waves in shallow water. Ann. New York Acad. Sci. **51**, 360–375 (1949).

***Stoker, J. J. The breaking of waves in shallow water.** Proc. Symposia Appl. Math., Vol. I, pp. 145–147. American Mathematical Society, New York, N. Y., 1949. $5.25.

For a fuller account cf. Communications on Appl. Math. **1**, 1–87 (1948); these Rev. **9**, 479.

Sekerž-Zen'kovič, Ya. I. On the theory of standing waves of finite amplitude on the surface of a heavy fluid. Doklady Akad. Nauk SSSR (N.S.) **58**, 551–553 (1947). (Russian)

In order to investigate the exact form of the waves mentioned in the title, the author, using Lagrangian coordinates, introduces a parameter into the functions describing the motion of the particles. These functions, which must satisfy appropriate differential equations and boundary conditions, are holomorphic in the parameter and uniquely determined for sufficiently small values of the parameter. Also, a velocity potential exists for the flow. The author next expands his functions in powers of the parameter, keeping terms through the fourth power. From this he is able to obtain approximately the shape of the surface wave, which he finds to be close to a trochoid. He also finds that there are no truly fixed nodes, the nodal points oscillating back and forth. [Reproduced (by permission of the editors) from Applied Mechanics Reviews.] *J. V. Wehausen.*

Sretenskiĭ, L. N. On annular waves on the surface of a rotating fluid. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 5–18 (1949). (Russian)

Periodic annular waves upon the surface of a fluid rotating in a circular cylinder of infinite depth are considered for two cases. In the first case the fluid is rotating as a rigid body, giving a paraboloid as equilibrium surface. The treatment is restricted to small waves and small angular velocities. It is found that the effect of the rotation is to increase the frequencies over those for a nonrotating fluid. The second case considered is one in which the axis is a vortex with circulation Γ (velocity of a particle at distance r is $\Gamma/2r$). To avoid the singularity on the axis the fluid is enclosed between two concentric cylinders. It is found again that the frequency is increased over that for a nonrotating fluid but that the amount of increase decreases as the ratio of the radius of the outer to that of the inner cylinder increases. *J. V. Wehausen* (Falls Church, Va.).

Honma, Masashi. On the calculation of flow containing jet. J. Soc. Appl. Mech. Japan **1**, 130–132 (1948). (Japanese. English summary)

When we calculate the equation of flow by a numerical method, the calculation must proceed in the upward direction from the lower end of the flow in the case of ordinary flow, that is, when the flow is lower than the wave in the channel. In the case of jet flow, that is, when the flow is faster than the wave, the calculation must proceed in the downward direction. In some problems it is possible for the state of flow to change from ordinary to jet, and then the initial point of calculation cannot be prescribed in either way. The author explains when such cases will happen and how the calculation is to be done in such cases. *From the author's summary.*

Burgers, J. M. Damped oscillations of a spherical mass of an elastic fluid. Nederl. Akad. Wetensch., Proc. **51**, 1211–1221 (1948). (English. Esperanto summary)

In order to interpret experiments by B. de Jong [de Jong and v. d. Berg, same vol., 1197–1210 (1948)], and using Lamb's theories [Hydrodynamics, 6th ed., Cambridge University Press, 1932, pp. 637–642, and references there given], the author investigates the oscillatory motions of a fluid sphere in which in spherical polars r, θ, ϕ the velocity is entirely in the ϕ direction and depends only on r, and

also those in which it is in the r and θ directions and depends on r and θ; when the damping is purely viscous, purely elastic, elastic with a relaxation time, or a combination of the latter with viscous effects; allowance for slip at the spherical boundary being made by assuming a coefficient of friction. The periods and logarithmic decrements and maximum shear stress are calculated. *M. J. Lighthill.*

Dobroklonskiĭ, S. V. Turbulent viscosity in the surface layer of the ocean and swell. Doklady Akad. Nauk SSSR (N.S.) **58**, 1345–1348 (1947). (Russian)

Štokman, V. B. Connections between the wind field, the complete current field and the average mass field in an inhomogeneous ocean. Doklady Akad. Nauk SSSR (N.S.) **59**, 675–678 (1948). (Russian)

Štokman, V. B. The influence of the profile of the bottom on the direction of the mean current excited by the wind or the field of a mass in an inhomogeneous ocean. Doklady Akad. Nauk SSSR (N.S.) **59**, 889–892 (1948). (Russian)

Golubeva, O. V. On simplifying the equations of hydrodynamics for studying surface currents in oceans. Doklady Akad. Nauk SSSR (N.S.) **61**, 453–456 (1948). (Russian)

Grad, Harold. Note on straight pipe jet motors. Comm. Pure Appl. Math. **2**, 71–77 (1949).

Grad, Harold. Resonance burning in rocket motors. Comm. Pure Appl. Math. **2**, 79–102 (1949).

Elasticity, Plasticity

Gleyzal, A. A mathematical formulation of the general continuous deformation problem. Quart. Appl. Math. **6**, 429–437 (1949).

The purpose of this paper is to formulate the general stress-strain problem for continuous flow or continuous deformation of continuous media. Embedded or comoving coordinates are used. The state of strain (in general finite) is determined by the metric tensor g_{ij} of the strained state, and it is assumed that the stress σ_{ij} is a tensor functional of g_{ij}, of tensors formed by differentiating or integrating g_{ij} with respect to a parameter t, and of the coordinates and t. The presence of t does not indicate a dynamical problem: t is a parameter which yields a sequence of equilibrium states. The equations of equilibrium for σ_{ij} may be used to obtain partial differential equations for three displacement functions. Two examples are discussed:

$$(1)\quad \sigma_{ij}=A\left[g_{ir}\int_0^t g^{rs}\dot{g}_{sj}dt+g_{jr}\int_0^t g^{rs}\dot{g}_{si}dt\right]+Bg_{ij}\int_0^t g^{mn}\dot{g}_{mn}dt$$

(A, B constant), which is described as the problem of a body under uniform hydrostatic pressure for the case of finite strain; (2) the tensile test, where the displacement functions are $\bar{x}^1=a(t)x^1$, $\bar{x}^2=b(t)x^2$, $\bar{x}^3=b(t)x^3$. The author hopes to give later a general theory of continuous deformations in which the notion of energy (not used in the present paper) is fundamental. [Reference might be made to the use of comoving coordinates by L. Brillouin, Les Tenseurs en Mécanique et en Élasticité, Masson, Paris, 1938, p. 226, and to the more general method of F. D. Murnaghan, Amer. J. Math. **59**, 235–260 (1937).] *J. L. Synge.*

Krečmer, V. V. On some problems of the theory of mechanical similarity. Doklady Akad. Nauk SSSR (N.S.) **65**, 457–460 (1949). (Russian)

Consider a pattern P and a model M and suppose that both are homogeneous isotropic elastic bodies, which are geometrically similar and are made of the same material. Let G denote the shear modulus and ν Poisson's ratio, both of which have the same values for P and M; and let the linear dimensions of P and M be in the ratio $\lambda:1=$constant. Let further the vectors of elastic displacements at corresponding points (x_p, y_p, z_p) and (x_m, y_m, z_m) of P and M be denoted by $\mathbf{u}_p$ and $\mathbf{u}_m$. If the forces acting on P and M are only surface forces, which are such that at corresponding points (1) $x_p=\lambda x_m$, $y_p=\lambda y_m$, $z_p=\lambda z_m$, the condition (2) $\mathbf{u}_p=\mathbf{u}_m$ is satisfied, then the Navier equations of elastic equilibrium of P and M are the same with corresponding boundary conditions. In this case the stress tensors and deformation tensors at corresponding points of P and M are equivalent. If on the contrary the forces acting are body forces, for example, those due to gravity, i.e., if the states of stress of P and M are produced by gravitational forces and $\lambda\neq 1$, then the condition (2) is not satisfied, and therefore in this case the stress tensors and deformation tensors at corresponding points of P and M are not equivalent.

There follows a description of a method by means of which, by introduction of artificial inertia forces, especially of centrifugal inertia forces, the statical similarity of P and M can be realized at least approximately [cf. Phillips, C. R. Acad. Sci. Paris **68**, 75–79 (1869); Kirpičev, Resistance of Materials, part 2, 1922]. By modeling with the aid of the centrifugal adjustment it is, however, impossible to realize either the dynamical or the relative dynamical similarity, i.e., when the vectorial relations between the corresponding quantities are not satisfied, but only the scalar relations, referred to different coordinate systems. It is shown by an example that in special cases it might be useful to apply the criterion of partial relative similarity, based on the consideration not only of the kinematical and dynamical elements, but also of the elements determined by elastic forces, which are produced by suddenly applied loads. It is supposed that the loads are geometrically similar and are made of the same material and that the elastic deformations are propagating through the bodies instantaneously. The relations obtained are analogous to that given by the criterion of similarity of Cauchy. *E. Leimanis.*

Moriguti, Sigeiti. On Castigliano's theorem in three-dimensional elastostatics. J. Soc. Appl. Mech. Japan **1**, 175–180 (1948). (Japanese. English summary)

Greenberg, H. J. Complementary minimum principles for an elastic-plastic material. Quart. Appl. Math. **7**, 85–95 (1949).

The author considers bodies consisting of an isotropic elastic-plastic material which obeys the stress-strain law of Prandtl-Rouss, and assumes also that the material is incompressible in both the elastic and plastic ranges. For such bodies, two minimum principles are formulated, relating to the mixed boundary-value problem where the velocities are given on part of the surface and the rate of change of the surface stresses on the remainder. The first principle, for

stress rates, is analogous to that of Castigliano in the theory of elasticity, while the second, for strain rates, corresponds to the principle of minimum potential energy. Relationships between these principles, and comparisons with previously formulated extremum principles in plasticity, are also pointed out. *F. B. Hildebrand* (Cambridge, Mass.).

***v. Mises, R. Three remarks on the theory of the ideal plastic body.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 415–429. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The author considers three problems concerning ideally plastic bodies in equilibrium. The stress function for torsion of noncircular cylinders is discussed when both elastic and plastic domains occur. From known elastic solutions, which are used for the elastic core of the shaft, a method of building on a plastic solution and finding the corresponding shaft boundary is given. The plane stress problem of a circular plate with a concentric circular hole under internal pressure is analysed. Ranges of pressure for elastic-plastic solutions are considered in terms of the radius ratio.

The general problem of plane stress is treated in terms of the principal stresses in the plane, writing each in terms of a single stress variable s by means of the flow condition. This, with θ, the angle giving the orientation of the principal axes, determines the stress tensor at each point. By analogy with the Chaplygin transformation in fluid mechanics, the equilibrium equations are transformed into a pair of linear equations with s and θ as independent variables. The properties of these equations and their characteristics in the s, θ "stress plane" are given for the Tresca maximum shear stress criterion, the Mises limit, and a new parabolic limit which consists of two parabolic arcs approximating to the other laws. *E. H. Lee* (Providence, R. I.).

***Prager, William. Discontinuous solutions in the theory of plasticity.** Proc. Symposia Appl. Math., Vol. I, pp. 211–212. American Mathematical Society, New York, N. Y., 1949. $5.25.

A fuller account appeared in the Courant Anniversary Volume, pp. 289–300, Interscience Publishers, New York, 1948; these Rev. **10**, 82.

Batdorf, S. B., and Budiansky, Bernard. A mathematical theory of plasticity based on the concept of slip. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1871**, 33 pp. (1949).

This theory expresses the relationship between stress and strain in plastic flow by representing the strain as the integrated sum of components based on a shear stress-slip relationship for all possible planes and directions of slip at a point. Thus the strain is obtained by double integration over all possible planes and directions of slip. The slip relationship for each plane and direction is given by a series expansion in terms of the over-stress above the yield stress in shear, and the determination of the coefficients when five terms are used is given in detail. A semi-graphical means of evaluating the integrals is developed. Comparison with an experiment in which axial compression of a tube was followed by twisting is shown to give satisfactory agreement. The authors show that the simplest previous theories would give poor agreement, but the reviewer believes that the more general theories which have been suggested [e.g., Prager, J. Appl. Phys. **20**, 235–241 (1949); these Rev. **10**, 495] would overcome the difficulties discussed. However, the slip theory does differ fundamentally from these others, and more critical experiments are required to arbitrate between them. *E. H. Lee* (Providence, R. I.).

***Nádai, A. Theory of the plastic distortion of thick-walled cylinders.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 430–448. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The stress and strain distributions are considered in long thick-walled circular cylinders deforming plastically under internal or external pressure combined with axial tension or compression. Elastic strains are neglected so that the deformation is incompressible. For ideally plastic material von Mises's quadratic limit is adopted, and for infinitesimal strains a deformation type of stress-strain relationship is used, determining strain components in terms of the stress. An incremental type of law is used for finite distortions, giving the strain rates as similar functions of the stress. With this particular formulation of the laws of plasticity there is a close correspondence between solutions for infinitesimal strains and for finite strains. Integration of the equilibrium equation for these conditions is carried out in terms of a transformation of independent variables which simplifies the analysis. The theory is extended to a work hardening material, retaining the incremental type of relationship between the ratios of the components of strain rate and stress, but requiring a functional relationship between the octahedral shearing stress and shearing strain analogous to a deformation type law. The stability of the radially symmetrical flow pattern is investigated. *E. H. Lee.*

Shepherd, W. M. Plastic stress-strain relations. Inst. Mech. Engrs. Proc. **159**, 95–99; discussion, 99–114 (1948).

The author suggests the following family of stress-strain laws for a strain-hardening elastic-plastic material:

$$(1)\quad d\epsilon_{ij}=E^{-1}[(1+\nu)d\sigma_{ij}-\nu\sigma_{kk}\delta_{ij}] +\tfrac{1}{2}f(S)(\sigma_{ij}-\tfrac{1}{3}\sigma_{kk}\delta_{ij})(dS+|dS|),$$

where $d\epsilon_{ij}$ is the strain increment produced by the stress increment $d\sigma_{ij}$, E denotes Young's modulus, ν Poisson's ratio, σ_{ij} the existing stress, δ_{ij} the Kronecker delta, and S the "equivalent tensile stress," i.e., an invariant of the stress tensor which, in the case of simple tension, reduces to the tensile stress. The laws of this family differ in the formula by which S is defined. The author shows that experiments of J. L. M. Morrison with thin-walled steel tubes under combined tension and torsion agree remarkably well with the particular stress-strain law which is obtained when the equivalent tensile test is defined in accordance with the yield condition of von Mises. [This particular law is a special case of the family of stress-strain laws discussed by G. H. Handelman, C. C. Lin, and W. Prager, Quart. Appl. Math. **4**, 397–407 (1947); these Rev. **9**, 120.] It is shown that, for each stress-strain law of the family (1), the (nonrecoverable) plastic work depends only on the final value of the equivalent tensile stress and not on the stress history.

The majority of the numerous contributions to the discussion of the paper are concerned with questions of experimental technique, relevance of experimental results, etc. In a theoretical contribution to the discussion W. P. Cashmore proposes to generalize the stress-strain law (1) by adding a term of the form $g(S)(d\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij})$ on the right-hand side as long as $dS\geqq 0$. Whereas (1) would not predict

any change in permanent strain for a neutral change of state (variation of the stresses such that the equivalent tensile stress remains constant), the modified stress-strain law might be made to represent changes of permanent strain if such were observed for neutral changes of state.

W. Prager (Providence, R. I.).

Gol'denblat, I. V. On the equations of equilibrium for a plastic medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 113–114 (1949). (Russian)

The author derives a stress-strain law from the assumption that the free energy is a function of the absolute temperature and the linear and quadratic invariants of the strain tensor. The resulting stress-strain law is of the deformation type. [Stress-strain laws of this type represent (nonlinear) elastic rather than plastic behavior.]

W. Prager (Providence, R. I.).

Fisher, J. C. Anisotropic plastic flow. Trans. A.S.M.E. **71**, 349–356 (1949).

Stress-strain relations for plastic deformation of isotropic and anisotropic metals are derived on the basis of a large number of assumptions. The basic one is that during slip over an area A, elastic energy is relieved in a volume equal to that of a sphere with great circle A, and the plastic strains observed are a summation of elastic strains relaxed in the neighborhood of a large number of slip bands. Incremental stress-strain relations are taken in the form $d\epsilon_{ij}=\lambda\partial W^*/\partial\sigma_{ij}$, where W^* is the difference between the strain energy per unit volume before and after slip. The von Mises form is obtained for isotropic polycrystals on the further assumption that the slip process relieves all shear stresses in the slip region, leaving the stress system hydrostatic. A simple anisotropic form requires taking the stresses after slip to be linear functions of the applied stresses.

D. C. Drucker (Providence, R. I.).

Hill, R. The plastic yielding of notched bars under tension. Quart. J. Mech. Appl. Math. **2**, 40–52 (1949).

The author discusses the state of stress existing in the core of a notched bar when pronounced plastic yielding begins. The material is assumed to be rigid as long as the stresses do not reach the yield limit, and plane plastic flow is assumed to develop once the yield limit is reached. After a general discussion of the problem the case of a deep notch with a semi-circular root is treated in detail. The relation of this problem to that of indentation with a flat punch is discussed.

W. Prager (Providence, R. I.).

Sokolovskiĭ, V. V. A plane problem of the theory of plasticity on the distribution of stresses around an opening. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 159–164 (1949). (Russian)

A method of finding the stresses in the plastic zones in the neighborhood of a hole was given by S. A. Christianovich for the case of plane strain, and by the present author for the case of plane stress. This method leads to boundary value problems for a system of equations of the hyperbolic type and the solution is found numerically. The present paper is devoted to another method for determining the stresses around a hole for plane stress or plane strain. This method uses trigonometric series.

From the author's summary.

Pacella, G. B. Su una proprietà della meccanica dei corpi continui e una deduzione geometrica della legge di Hooke. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 31–37 (1948).

Some simple geometrical considerations about the two systems of cones generated by the normals to planes across which (a) normal stress is constant, (b) normal strain is constant. Hooke's law for an isotropic body implies that these two systems of cones are the same; the converse theorem involves an assumption of linearity.

J. L. Synge (Dublin).

Green, A. E. On Boussinesq's problem and penny-shaped cracks. Proc. Cambridge Philos. Soc. **45**, 251–257 (1949).

Using a solution of an integral equation developed by Copson in connection with the problem of the electrified disc [Proc. Edinburgh Math. Soc. (2) **8**, 14–19 (1947); these Rev. **9**, 318], the author solves a number of "crack" and "punch" problems in elasticity. Most of these problems have been worked out by the solution of other types of integral equations. [See, for example, Harding and Sneddon, same Proc. **41**, 16–26 (1945); these Rev. **6**, 251.]

A. E. Heins (Pittsburgh, Pa.).

Willmore, T. J. The distribution of stress in the neighbourhood of a crack. Quart. J. Mech. Appl. Math. **2**, 53–63 (1949).

The stress distribution round a linear crack in plane strain is obtained using complex variable analysis based on Muschelišvili's method [Z. Angew. Math. Mech. **13**, 264–282 (1933)]. The stress function is given for an internal pressure distribution in the form of the terms of Fourier series, thus determining the solution for an arbitrary pressure distribution. The solution is generalised for aelotropic materials possessing two directions of elastic symmetry, one being parallel to the crack. The case of two collinear cracks is solved by means of a hydrodynamical analogy, using a known corresponding complex potential function. The stress distribution is discussed and also its application to Griffith's fracture theory [Philos. Trans. Roy. Soc. London. Ser. A. **221**, 163–198 (1920)].

E. H. Lee (Providence, R. I.).

Reissner, H., and Morduchow, M. Reinforced circular cutouts in plane sheets. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1852**, 60 pp. (1949).

The problem dealt with is that of designing a reinforcing ring for the purpose of reducing the stress concentration in the vicinity of a circular cutout in a flat sheet; loads are applied to the sheet so as to induce only a condition of plane stress in the system. The Airy stress function for the sheet is taken in general form, suitable for the case where the edge stresses of the cutout have zero resultant, and the reinforcing ring is treated by the conventional theory for thin curved bars. The problem is then approached from two directions: (a) for various sheet loadings, a study is made of the possibility of designing a reinforcement which entirely eliminates any stress disturbances due to the cutout; (b) where this is not possible, the stress reductions corresponding to rings of uniform cross-section are calculated. The sheet loadings considered are applied distant from the cutout, and consist of hydrostatic stress, pure shear, uni-axial tension and pure bending. Only the first two cases permit complete removal of the cutout disturbances by means of a

reinforcing ring. For the case of uni-axial tension, an experimental check of the theoretical results for a single configuration was made. The agreement between the test and theory is fair, and reasons are advanced for the differences encountered.

Problems of this kind have previously been studied by numerous investigators; for example, see Levy, McPherson, and Smith [J. Appl. Mech. **15**, 160–168 (1948); a further bibliography is contained in that paper]. *M. Goland.*

Hank, R. J., and Scrivner, F. H. Stresses and displacements in a semi-infinite elastic body with parabolic cross section acted on by its own weight only. J. Appl. Mech. **16**, 211–212 (1949).

M'Ewen, Ewen. Stresses in elastic cylinders in contact along a generatrix (including the effect of tangential friction). Philos. Mag. (7) **40**, 454–459 (1949).

The elastic stress distribution along the line of contact of parallel circular cylinders is obtained by representing the stress components in plane strain in terms of an appropriate complex stress function. The particular choice is shown to satisfy the contact conditions for incipient circumferential slip, with constant coefficient of friction μ, if the cylinders are of the same material, and to reduce to Hertz's solution [J. Reine Angew. Math. **92**, 156–171 (1882)] for $\mu=0$, without the restriction of the same material.

E. H. Lee (Providence, R. I.).

Rivlin, R. S. Large elastic deformations of isotropic materials. V. The problem of flexure. Proc. Roy. Soc. London. Ser. A. **195**, 463–473 (1949).

As a further application of the method developed in a previous paper [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 379–397 (1948); these Rev. **10**, 340], the author discusses the flexure of a cuboid of highly elastic incompressible material for which the stored-energy is a function of the strain invariants. He assumes a very simple type of displacement in which the cross section and two of the sides remain plane, the other two sides becoming circular cylinders. It is found that the resulting strain can be maintained without body forces by the application of suitable surface tractions. Using a stored-energy function employed earlier [Proc. Cambridge Philos. Soc. **44**, 595–597 (1948); these Rev. **10**, 216] the author succeeds in getting rid of the surface tractions on the curved surfaces, but those on the plane sides remain. The tractions on the cross sections are equivalent to couples. *J. L. Synge* (Dublin).

Rivlin, R. S. A note on the torsion of an incompressible highly-elastic cylinder. Proc. Cambridge Philos. Soc. **45**, 485–487 (1949).

In a series of earlier papers [Philos. Trans. Roy. Soc. London. Ser. A. **240**, 459–490, 491–508, 509–525 (1948); **241**, 379–397 (1948); these Rev. **10**, 168, 340] the author studied large elastic deformations of incompressible isotropic materials. In this paper some simplifications of the results of an earlier paper are made in the expressions for the end tractions necessary to maintain a cylinder of this material in a state of simple torsion. The resultant normal surface traction and the resultant end couple are expressed in terms of the energy function W, the latter containing only the first two invariants I_1 and I_2. *D. L. Holl* (Ames, Iowa).

Morris, Rosa M. Some general solutions of St. Venant's torsion and flexure problem. III. Proc. London Math. Soc. (2) **51**, 424–439 (1949).

In parts I and II [same Proc. (2) **46**, 81–98 (1940); **49**, 1–18 (1945); these Rev. **1**, 189; **7**, 501] the author considered the torsion and flexure of uniform beams the cross section of which could be mapped by

$$z=\sum_{0}^{\infty}a_n e^{(n-1)\zeta}$$

on a strip in the $\zeta=(\xi+i\eta)$-plane. The present paper extends the results to include hollow cylinders whose section is bounded by two of the net surfaces $\eta=\alpha_1$, and $\eta=\alpha_2$. Complex potentials $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ are found in series form satisfying the required boundary conditions of the flexure problem. Applications are made to annuli bounded by two confocal ellipses, two eccentric circles, and two cardioids. The center of flexure is found in the two latter cases.

D. L. Holl (Ames, Iowa).

Mitra, D. N. On the flexure problem for some boundaries. Bull. Calcutta Math. Soc. **40**, 173–182 (1948).

This paper contains the solution of the pure flexure problem for three boundaries: (1) the inverse of an ellipse with respect to its center, (2) two orthogonal circular arcs of equal radii, (3) a loop of the lemniscate of Bernoulli. The solution for the third boundary was obtained in a different way by A. C. Stevenson [Philos. Trans. Roy. Soc. London. Ser. A. **237**, 161–229 (1938)]. The author employs the general method proposed by S. Ghosh [same Bull. **39**, 1–14 (1947); these Rev. **9**, 256] for the solution of the pure flexure problem. *J. B. Diaz* (Providence, R. I.).

Pizzetti, Giulio. Contributo allo studio del problema di De Saint-Venant in campo elasto-plastico. Pont. Acad. Sci. Comment. **11**, 1–30 (1947).

The author applies Colonnetti's principle [Pont. Acad. Sci. Comment. **2**, 131–150, 309–319, 439–514 (1938)] to the analysis of stresses and strains in elastic-plastic bars. The following cases are treated in detail: pure flexure (rectangular cross section), torsion (circular cross section), bending by transverse forces (rectangular cross section), combined flexure and axial compression (rectangular cross section).

W. Prager (Providence, R. I.).

Šerman, D. I. On the state of stress in some shrink-fitted members. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1948**, 1371–1388 (1948). (Russian)

The paper contains a solution of the following two-dimensional elastic problem. A long hollow prismatic body whose section by a plane normal to the axis of the prism is a square (with rounded corners) with a circular hole at the center of the square, is shrink-fitted on a solid circular shaft. The elastic properties of the shaft are identical with those of the prism, and the lateral surface of the prism is free of stress. What is the state of stress in the member so formed? If the boundary of the square in the (x, y)-plane is L, and that of the circular hole is γ, the solution of the problem, following Muschelišvili, reduces to the search for four functions $\varphi_j(z)$ and $\psi_j(z)$ $(j=1, 2)$ of a complex variable $z=x+iy$, analytic in the regions S_j, where S_1 is a doubly-connected region bounded by L and γ and S_2 is the circular region bounded by γ.

The functions φ_i and ψ_i are determined by the following boundary conditions:

$$\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}=0$$

on L;

$$\varphi_1(t)+t\overline{\varphi_1'(t)}+\overline{\psi_1(t)}=\varphi_2(t)+t\overline{\varphi_2'(t)}+\overline{\psi_2(t)},$$
$$\kappa\varphi_1(t)-t\overline{\varphi_1'(t)}-\overline{\psi_1(t)}=\kappa\varphi_2(t)-t\overline{\varphi_2'(t)}-\overline{\psi_2(t)}+2kt$$

on γ, where κ and k are constants determined by the elastic properties of the medium and by the amount of shrink along γ. The author reduces the problem (by means of analytic continuation) to the determination of only two functions $\varphi(z)$ and $\psi(z)$, analytic in the region S_1+S_2, of the form

$$\varphi(z)=\frac{1}{2\pi i}\int_L\frac{\omega(t)}{t-z}dt;\quad \psi(z)=\frac{1}{2\pi i}\int_L\frac{\overline{\omega(t)}-\overline{t\omega'(t)}}{t-z}dt,$$

where $\omega(t)$ satisfies a certain integral-differential equation and bars denote conjugate values. The solution of the latter equation is obtained in the form of an infinite series. A calculation of the distribution of normal stress along γ, contained in the paper, illustrates the practical value of the function-theoretic methods of solution of elastic problems.

I. S. Sokolnikoff (Los Angeles, Calif.).

Šerman, D. I. On a problem of torsion. Doklady Akad. Nauk SSSR (N.S.) **63**, 499–502 (1948). (Russian)

The author gives an approximate solution of the torsion problem for a hollow cylinder whose exterior boundary is a circle L_1, and the interior an ellipse L_2, with coincident centers. The usual complex torsion function $\varphi_1(z)$ is shown to be related to a function $\varphi(z)$, analytic in the region exterior to L_2, and vanishing at infinity. This latter function is determined by the method used by the author in investigating the state of stress in shrink-fitted members [see the preceding review]. Numerical calculations, performed for the case when the semi-axes a and b of the ellipse are such that $a=3b$, and the radius of the circle $R=4b$, indicate the effectiveness of the approximate solution.

I. S. Sokolnikoff (Los Angeles, Calif.).

Arutyunyan, N. H. Solution of the problem of the torsion of a rod with a polygonal cross section. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 107–112 (1949). (Russian)

An exact solution of the torsion problem for an L-shaped beam is obtained in the form of a doubly-infinite series involving exponential and trigonometric functions. The values of torsional rigidities and shearing stresses are calculated for several L-shaped beams whose sides are of equal lengths but of different widths. *I. S. Sokolnikoff.*

Birman, S. E. Solutions for thin-walled bars by means of polynomials. Doklady Akad. Nauk SSSR (N.S.) **65**, 283–286 (1949). (Russian)

The note contains an illustration of the use of polynomials [see same Doklady **62**, 187–190, 305–308 (1948); these Rev. **10**, 341] in calculating stresses in a thin-walled hollow rectangular cantilever beam subjected to a special form of linear loading. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Eddy, R. P., and Shaw, F. S. Numerical solution of elasto-plastic torsion of a shaft of rotational symmetry. J. Appl. Mech. **16**, 139–148 (1949).

Hadji-Argyris, J., and Dunne, P. C. The general theory of cylindrical and conical tubes under torsion and bending loads. Single and many cell tubes of arbitrary cross-section with rigid diaphrams. VI. J. Roy. Aeronaut. Soc. **53**, 461–483, 558–620 (1949).

The first four parts appeared in the same J. **51**, 199–269 (1947); part V, ibid., 757–784, 884–930 (1947); these Rev. **8**, 613; **9**, 122, 256.

Lavrent'ev, M. A., and Išlinskiĭ, A. Yu. The dynamical forms of the loss of stability of elastic systems. Doklady Akad. Nauk SSSR (N.S.) **64**, 779–782 (1949). (Russian)

The number of forms of unstable equilibrium of a rod and tube under compressive forces are computed and tabulated. *T. C. Doyle* (Hanover, N. H.).

Krzywoblocki, M. Zbigniew. A general approximation method in the theory of plates of small deflection. Quart. Appl. Math. **6**, 31–52 (1948).

Krzywoblocki, M. Z. A general approximation method in the theory of plates of small deflection. Quart. Appl. Math. **7**, 236 (1949).

The analysis of small deflections of thin elastic plates is considered by means of Bergman's operator method [Duke Math. J. **11**, 617–649 (1944); these Rev. **6**, 69], which is applicable to plates of arbitrary shape. These problems involve the solution of linear partial differential equations of the fourth order with variable coefficients. The method of solution is discussed, representing the required solution as a series of functions, which are chosen to satisfy the boundary conditions approximately. Problem types in the theory of plates are discussed, and, for the application of this method, those resulting in nonhomogeneous equations must be reduced to the homogeneous form. A particular example of the buckling of a triangular plate under a uniform compressive stress parallel to one side is considered. The boundary condition of clamped edges is satisfied approximately in terms of a variational problem. Sets of solutions for a range of values of the compressive stress are considered, the first critical stress being obtained from an appropriate choice from these. Application of this method involves a large number of simple numerical operations, and punch-card machines were used.

In the second note the author points out that the methods of Weinstein [Étude des spectres des équations aux dérivées partielles de la théorie des plaques élastiques, Mémor. Sci. Math., no. 88, Gauthier-Villars, Paris, 1937] and Southwell [Relaxation Methods in Theoretical Physics, Oxford, 1946; these Rev. **8**, 355] also apply to such types of problems.

E. H. Lee (Providence, R. I.).

Mann, E. H. Shearing displacement of a rectangular plate. Proc. Cambridge Philos. Soc. **45**, 258–262 (1949).

The problem dealt with is one of plane stress in elastic theory: to find the stresses in a thin flat rectangular plate when two opposite edges are displaced a given distance relative and parallel to one another, the other two opposite edges being left free. The problem has already been solved by C. E. Inglis [Proc. Roy. Soc. London. Ser. A. **103**, 598–610 (1923)], but the present author provides a more rapid and convenient way for arriving at the results. Both authors establish a solution in terms of infinite series, which are sufficiently rapidly convergent so that only the first terms are practically important. The series solution used by the present author automatically satisfies two of the boundary

conditions, i.e., constant displacement of the two restrained edges relative to each other, and zero normal stress on the free edges. The remaining boundary conditions are satisfied by adjusting the coefficients of the series. Particularly simple means for arriving at the coefficients are deduced, not involving the solution of simultaneous equations. The present solution, using only four terms of the infinite series, arrives at results comparable in accuracy with those deduced by Inglis with twenty-two terms of his series taken into account. *M. Goland* (Kansas City, Mo.).

Reutter, F. Halbebene und Parallelstreifen mit veränderlichem Elastizitätsmodul. Ing.-Arch. 16, 307–320 (1948).

The stress distributions in a semi-infinite body with surface $x=0$, and in an infinite plate with surfaces $x=\pm b$, are considered, when the elastic modulus E is a function of x. A body force in the x direction, depending on x, is included. Such solutions are of interest in soil mechanics, and in the theory of sandwich plates which are stronger if the lightweight filler between the metal faces is more rigid close to the faces. A sinusoidal distribution of normal surface stress $P_0 \sin my$ with only normal displacements is considered, this problem being suggested in connection with the instability of sandwich plates. In the case of the plate, both symmetrical and anti-symmetrical loading on the two faces are treated. Solutions are obtained for quadratic, cosh, and exponential variations of E, some solutions being expressible in closed form, others as infinite series. Most of the solutions assume Poisson's ratio $\nu=0$, but it is shown, by considering a special case, that this has comparatively little influence on the stress distribution. *E. H. Lee* (Providence, R. I.).

Sternberg, E., and Sadowsky, M. A. Three-dimensional solution for the stress concentration around a circular hole in a plate of arbitrary thickness. J. Appl. Mech. 16, 27–38 (1949).

A modified Ritz method is employed to calculate the three-dimensional state of stress in a homogeneous isotropic infinite plate with a completely stress-free circular hole. The boundary conditions at infinity are taken as uniform plane stress parallel to the bounding planes. Numerical values and graphs of the interesting stress components are given for ratios of thickness of plate to diameter of hole from 0 to 50. Change in plate thickness is calculated also. Solutions are obtained for the basic problems of a plane hydrostatic state of stress and a plane state of pure shearing stress at infinity. The authors use series expressions for three new stress functions to minimize the strain energy for the residual problem of plane stress (the quadratic variation which adds to zero in the axial direction over the circular boundary). The process is carried out successfully with three and with six parameters and the influence of Poisson's ratio is shown. Stress concentration factors apparently vary little from those obtained by the usual two-dimensional analysis. Plane strain values for the transverse normal stress are practically fully developed at the middle plane when the ratio of thickness to diameter exceeds two.

D. C. Drucker (Providence, R. I.).

Hayashi, Gorô. State of stress in a moderately thick plate. Tôhoku Math. J. 48, 257–271 (1941).

In the first part of the paper there are reproduced some known exact solutions for a plate with both faces free of stress. In the second part the author obtains some results for a plate with normal stress applied to the faces of the plate. These results are related to work by S. Woinowsky-Krieger [Ing.-Arch. 4, 203–226, 305–331 (1933)].

E. Reissner (Cambridge, Mass.).

Aleck, B. J. Thermal stresses in a rectangular plate clamped along an edge. J. Appl. Mech. 16, 118–122 (1949).

Supino, Giulio. Le condizioni al contorno per le lastre sottili inflesse. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 2, 85–89 (1946).

The paper contains modifications of Kirchhoff's method of proof concerning the contraction of three boundary conditions into two conditions for the problem of transverse bending of elastic plates when formulated as a boundary value problem of the equation $\Delta\Delta w=p$. *E. Reissner.*

Conway, H. D. Note on the bending of circular plates of variable thickness. J. Appl. Mech. 16, 209–210 (1949).

Ikeda, K., Uemura, M., and Muto, M. On the plane stresses in anisotropic plates having a circular or elliptic hole with uniform pressure or shearing force on its periphery. J. Soc. Appl. Mech. Japan 1, 8–14 (1948). (Japanese. English summary)

The aim of the present paper is to show that the so-called Wolf's assumption concerning the elastic constants of the anisotropic plate is adequate for solid wooden plates but not for plywoods. Especially in such a problem as is treated here, considerable difference is found between the stress distributions of the two kinds of plates, and that the distribution in the raw wooden plate is approximately the same as that in the isotropic plate.

From the authors' summary.

***Stoker, J. J. Pre-stressing a plane circular plate to stiffen it against buckling.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 268–276. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

A circular plate is pre-stressed by cutting a radial slit and inserting a narrow wedge in the form of a circular sector of small central angle α. The edges of the wedge are then welded to the plate. It is shown that the buckling stresses under uniform radial compression applied to the circular edge of the plate may be as much as three times as great as those for the initially unstressed plate. The lowest critical stress increases with the angle α up to a certain value α_c, after which it begins to decrease. For example, for a plate for which the ratio of thickness to radius is 0.01 the angle α_c is less than one-half of a degree. The existence of stress concentration at the center of the plate is recognized but is considered not to present difficulties that cannot be overcome in practical cases. *H. W. March* (Madison, Wis.).

Grossman, Norman. Elastic stability of simply supported flat rectangular plates under critical combinations of transverse compression and longitudinal bending. J. Aeronaut. Sci. 16, 272–276 (1949).

The characteristic value problem for this case of plate buckling is reduced, by conventional application of the Rayleigh-Ritz energy procedure, to the condition of vanishing of an nth order determinant. Restricting attention to an approximation in which $n=4$ the characteristic values are then determined by a matrix iteration method. A reduction in labor is accomplished by separating the approxima-

tion into an odd and an even component. Various practically interesting properties of the solution are discussed in detail. *E. Reissner* (Cambridge, Mass.).

Milosavljevitch, Miodrag. Sur la stabilité des plaques rectangulaires renforcées par des raidisseurs et sollicitées à la flexion et au cisaillement. Acad. Serbe Sci. Publ. Inst. Math. **1**, 121–135 (1947).

A simple supported elastic rectangular plate with elastic stiffness lying parallel to both the transverse and longitudinal edges is subjected to loads which might produce buckling. The critical magnitude of this load is determined using a Fourier series technique. *G. F. Carrier.*

Libove, Charles, Ferdman, Saul, and Reusch, John J. Elastic buckling of a simply supported plate under a compressive stress that varies linearly in the direction of loading. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1891**, 33 pp. (1949).

Moriguchi, S. Apparent modulus of elasticity of a circular plate just after buckling. J. Soc. Appl. Mech. Japan **1**, 81–86 (1948). (Japanese. English summary)

An isotropic homogeneous flat circular plate of uniform thickness is loaded by pressure uniformly distributed over its boundary. It is supported on its boundary, so that no deflection can take place there, but its inclination at the boundary is either completely prescribed or free. The form of the buckled plate is assumed to have rotational symmetry. "Incompatibility" of the plate is either zero everywhere or uniformly distributed. The apparent modulus of elasticity just after buckling is calculated by the perturbation method. *From the author's summary.*

Yoshiki, Masao. A new method of determining the critical buckling points of rectangular plates in compression. J. Soc. Appl. Mech. Japan **1**, 193–199 (1948). (Japanese. English summary)

To overcome the difficulty of determining the critical buckling load of thin rectangular plates in compression, a new method is proposed, which is based upon the load-deflection characteristics of plates having small initial deformation at the post-buckling region. This method is applied to the determination of the buckling loads of thin rectangular plates in compression which are freely supported at four edges. It is found to be very satisfactory compared with the methods originated by Cox, Southwell and Donnell. *From the author's summary.*

Budiansky, Bernard, and Hu, Pai C. The Lagrangian multiplier method of finding upper and lower limits to critical stresses of clamped plates. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **848**, 11 pp. (1946).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1103** (1946); these Rev. **8**, 118.

identical with the paper on page 416.

Vlasov, V. S. Some new problems on shells and thin structures. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1204**, 46 pp. (1949).

This is a translation of a survey of the author's work published during the period 1933–1946 [cf. the following review]. Contributions are described to approximate theories of cylindrical shells of the type now employed in aircraft structural analysis and also to the classical theory of thin shells with or without resistance to bending. In the approximate theories certain presumably unimportant stress couples and resultants are neglected together with certain types of deformation. The resultant system of equations is reduced to one equation for a suitably chosen stress function. For this equation a characteristic function method of solution is outlined. It is next shown how the classical buckling and vibration problem of the small deflection theory can be reduced to a canonical system of two simultaneous differential equations. It is further mentioned that results have been obtained concerning the differences of the mathematical theory of shells without bending resistance (membranes) depending on whether the Gaussian curvature of the shell is positive or negative. *E. Reissner* (Cambridge, Mass.).

Vlasov, V. Z. Some new problems on the structural mechanics of shells and thin structures. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 27–52 (1947). (Russian)

A translation is reviewed above.

Reissner, Eric. Small bending and stretching of sandwich-type shells. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1832**, ii+89 pp. (1949).

This report contains a comprehensive analysis of the bending and stretching of sandwich-type shells, according to the small deflection theory. After developing the basic equations for a loaded shell of general character, the author specializes the system for the cases of flat plates, circular rings, circular cylindrical shells, and spherical shells with axisymmetrical deformations. The study does not embrace any considerations of elastic stability. The sandwich construction dealt with is presumed to have two identical face layers which are small in thickness compared with the core ($t/h \ll 1$); also, the face layer stiffness $E_f t$ is taken to be large compared with the core layer stiffness $E_c h$ (where E is the Young's modulus in each case). These assumptions lead to the conclusions that the face layers can be treated as loaded membranes (having negligible bending stiffness in their own right), while in the core only the transverse shear and normal stresses need be taken into account. This is equivalent to assuming that the core layer stresses contribute negligibly to the stress couples and tangential stress resultants in the composite shell, while the face layer stresses are unimportant in their contribution to the transverse shear stress resultants.

After a discussion of the stress distributions in the face and core layers, the equations governing the small bending and stretching of shells of general character are derived from energy considerations, making use of a theorem attributed to Trefftz [Z. Angew. Math. Mech. **15**, 101–108 (1935)]. On the basis of these equations, a number of specific problems are dealt with; these include a variety of loaded flat plate configurations, as well as problems pertaining to circular rings, circular cylindrical shells, and spherical shells with axisymmetrical deformations. In the flat plate studies, the effects of transverse shear deformation are studied; the importance of this factor in determining the behavior of flat sandwich-type plates is known from past work, and is investigated in further detail in the present analysis. In the case of sandwich shells, the transverse shear deformations continue to exert an important influence, and a new effect is found to be significant, namely, the effect of transverse normal stress deformation. This latter effect has not been considered in previous analyses of sandwich structures, and hence its appearance in this study

is of interest. It is of interest to note that general rules regarding the importance of the transverse shear and transverse normal stress deformations on the behavior of systems apparently cannot be formulated, since the magnitude of their effect depends not only on the nature of the sandwich construction, but also on the character of the loading and geometry of the structure. *M. Goland.*

Panferov, V. M. On the convergence of the method of elastic solutions in the theory of elastic-plastic deformations of shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 79–94 (1949). (Russian)

Applying his theory of plastic deformation to the axially symmetric deformation of cylindrical shells of an elastic-plastic material, A. A. Ilyushin [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 245–272 (1943); these Rev. **6**, 252; translated in Plastic Deformation, Principles and Theories, pp. 45–96, Brooklyn, N. Y., 1948; these Rev. **10**, 170] used a method of successive approximations which he called the method of elastic solutions. The present paper establishes the existence and uniqueness of the solution of Ilyushin's problem and the convergence of his method.
W. Prager (Providence, R. I.).

Pailloux, Henri. Sur la détermination des tensions dans une membrane dépourvue de raideur. C. R. Acad. Sci. Paris **228**, 54–56 (1949).

Using vector notation, the author obtains three partial differential equations of equilibrium satisfied by three quantities which specify the stress in a curved membrane without stiffness. [For equivalent tensor equations in the more general case of a shell, see J. L. Synge and W. Z. Chien, Theodore von Kármán Anniversary Volume, California Institute of Technology, Pasadena, Calif., 1941, pp. 103–120, in particular, p. 110; W. Z. Chien, Quart. Appl. Math. **1**, 297–327 (1944), in particular, p. 299; these Rev. **3**, 30; **5**, 195.] The deformation due to a small displacement of the membrane is discussed, and stress-strain relations are given.
J. L. Synge (Dublin).

Schunck, T. E. Der zylindrische Schalenstreifen oberhalb der Beulgrenze. Ing.-Arch. **16**, 403–421 (1948).

The author investigates the state of stress and deformation in slightly curved cylindrical panels of infinite length, acted upon by normal and shear loads. The basic equations are taken in the form given by K. Marguerre [Proc. 5th Internat. Congress Appl. Math. 1938, pp. 93–101 (1939)] and approximate solutions are obtained by the direct methods of the calculus of variations (energy methods). Extensive numerical results are obtained for buckling loads and for the stress distributions and deflection patterns due to loads beyond the buckling load. [It seems to the reviewer that the work of this paper should be extended by a consideration of the possibility of the shell's buckling before the classical buckling load has been reached, in accordance with the work by Donnell, Cox, von Kármán and Tsien [J. Aeronaut. Sci. **8**, 303–312 (1941); these Rev. **4**, 63] and others.] *E. Reissner* (Cambridge, Mass.).

Panov, D. Yu., and Feodos'ev, V. I. Letter to the editor. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 116 (1949). (Russian)

Attention is called to a missing term in equation (3.7) of the authors' paper on stability of shallow shells [Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 389–406 (1948); these Rev. **10**, 218]. The inclusion of this term does not alter the method of solution, but changes the values of the critical loads. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Astbury, N. F. Some theoretical considerations on the dynamic properties of plastics. Proc. Roy. Soc. London. Ser. A. **196**, 92–105 (1949).

The dynamic behavior of plastic materials is considered, using for the stress-strain relationship a linear relationship between stress, rate of change of stress, strain, and rate of change of strain. This corresponds to the response to a stress increment consisting of an instantaneous elastic strain followed by the development of additional plastic strain based on a constant relaxation time. The configuration of standing plane waves in such a medium is investigated, by expressing the displacement in complex form involving a wave velocity and an attenuation constant, and choosing these to satisfy the equation of motion and the stress-strain relationship. The reflection and transmission of waves from a steady source with normal incidence on a plane slab of such material is investigated in the same way. The attenuation is shown to have a marked influence on the magnitude of the reflected wave. *E. H. Lee* (Providence, R. I.).

Graffi, Dario. Sul teorema di reciprocità nella dinamica dei corpi elastici. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **4** (1946–47), 103–109 (1948).

The author is concerned with extension of Betti's theorem of reciprocity in the infinitesimal theory of elasticity to the case of small motions without damping. So as to be able to use the method of the Laplace transform, he supposes all functions and necessary derivatives occurring to be bounded continuous functions at all points and times. He then derives a reciprocal theorem which can be formulated in the following way. In a possible motion 1 let F_1 and R_1 be the extraneous and surface force vectors; S_1 and V_1 the displacement and velocity vectors; S_{10} and V_{10} the initial values of these latter; and let a subscript 2 denote the corresponding quantities in a second possible motion. Then for a homogeneous body of density ρ the expression

$$\int_0^t d\tau \int_{\mathfrak{B}} \rho F_1(t-\tau)\cdot S_2(\tau)\,d\mathfrak{B} + \int_0^t d\tau \oint_{\mathfrak{S}} R_1(t-\tau)\cdot S_2(\tau)\,d\mathfrak{S} + \int_{\mathfrak{B}} (S_2\cdot V_{01} + V_2\cdot S_{01})\,d\mathfrak{B}$$

is invariant under interchange of the subscripts 1 and 2. The author shows that this theorem includes as special cases his former results on this subject [Ann. Mat. Pura Appl. (4) **18**, 173–200 (1939)]. Among other special cases he notes also that in the case $R_1=R_2=0$, if $\rho F_1=G(t)a_1$, $S_{01}=\lambda a_1$, $V_{01}=\mu a_1$, where a_1 is a vector field, and similarly $\rho F_2=G(t)a_2$, etc., then Betti's theorem remains valid in the dynamical case. The form to which the general theorem immediately reduces in the case of free oscillations is also quite simple. *C. Truesdell* (Washington, D. C.).

Sauter, Fritz. Bemerkungen zur Schwingungstheorie dünner elastischer Platten. Z. Naturforschung **3a**, 548–552 (1948).

Basing his work on solutions previously obtained by Rayleigh [Proc. London Math. Soc. (1) **20**, 225–234 (1889)] and H. Lamb [Proc. Roy. Soc. London. Ser. A. **93**, 114–128 (1917)], the author presents a general discussion of the shear, extensional and flexural wave motions encountered in bars and plates. The author's purpose is to call these

results of Rayleigh and Lamb to the attention of present-day workers, since it appears they have been given insufficient attention in modern literature on the subject. The solutions in question are obtained by the methods of the theory of elasticity, and hold for unidirectional wave propagation by plane strain. They permit discussion of all cases ranging from "short" to "long" wave-lengths (as compared with the plate thickness). The "long" waves correspond to the configurations treated in the conventional, thin plate theory. The general problem is reviewed and the Rayleigh and Lamb results are interpreted. Also, attention is given to the physical reasons why the bending waves in the elementary theory must satisfy a fourth-order differential equation, while the shear and extension waves follow a second-order mechanism. *M. Goland.*

Pignedoli, Antonio. Sulle vibrazioni di una piastra circolare sollecitata al contorno da una pressione radiale pulsante. Atti Sem. Mat. Fis. Univ. Modena **2**, 3–19 (1948).

Dimentberg, F. M. On the transverse oscillations of a heavy bar when resistance is present. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 51–54 (1949). (Russian)

The author calculates the dynamic stiffness (mechanical impedance) of a bar when inner and outer friction are present. *W. H. Muller* (Amsterdam).

Takeyama, Hisao. On the strength and vibration of a rotating rod. I. J. Soc. Appl. Mech. Japan **1**, 121–129 (1948). (Japanese. English summary)

From the fundamental equations of the bending strength and lateral vibrations of rotating bars, integral equations of Volterra or Fredholm type have been derived by using the bending moment M or the inclination of deflection ϕ. These are easily solved numerically or analytically whether the number of rotations is small or not.

From the author's summary.

Takesi, Kanazawa. On the shock waves transmitted in conical bars. J. Soc. Appl. Mech. Japan **1**, 133–141 (1948). (Japanese. English summary)

Characteristics of longitudinal and transverse shock waves in conical bars were investigated analytically. The differences between the waves in conical bars and those in uniform bars were found. The main results are as follows. If there is no damping effect due to the viscosity of the materials, the waves are damped on account of the variation of the cross-sectional area. Longitudinal waves produced from a harmonic vibrating source will be totally reflected from a junction where the other materials are connected, if the density is very small in comparison with the original one, and their Young's moduli are nearly equal. If the viscosity of the materials is neglected, the amplitude of lateral progressive waves produced from a harmonic vibrating source is proportional to the $-\frac{1}{4}$ power of the distance from the vertex of the conical bar; and its wave length varies as the square root of the radius of the cross-section.

From the author's summary.

Ogibalov, P. M. On the duration of the visco-plastic flow of a substance after an impulsive load. Učenye Zapiski Moskov. Gos. Univ. Mehanika **122**, tom II, 183–192 (1948). (Russian)

A plastic material is considered which deforms in shear under the stress $\tau_{max}=\pm k+\mu v+h\tan\gamma$, the three terms being respectively the initial quasi-static yield stress, a viscous rate of strain term, and a linear work hardening term; elastic strains being neglected. A circular cylinder of rigid material is embedded in a plane slab of the plastic material, with its axis normal to the slab. Initially the cylinder is given an impulsive angular velocity ω_0, and subsequently moves under the influence only of the torque imposed by the plastic material. The inertia of the plastic material is neglected in comparison with that of the cylinder; this gives the stress distribution immediately, for $\tau_{r\theta}$ is the only non-vanishing stress component. Without this simplification the problem would be one of plastic wave propagation. A third order partial differential equation is obtained for the angular displacement in the plastic material, a solution of which is expressed in terms of three arbitrary functions. The boundary conditions determine a second order ordinary nonlinear differential equation, which is solved for the case of no work hardening. The extent of the region straining at any time is determined; and the time to bring the cylinder to rest.

E. H. Lee (Providence, R. I.).

Ogibalov, P. M., and Loginova, M. A. On the dependence of the strains in a rapid deformation under impulsive loading beyond the elastic limit. Vestnik Moskov. Univ. **3**, no. 5, 39–58 (1948). (Russian)

The analysis of a dynamic compression test is considered, in which a mass strikes a cylindrical specimen and compresses it axially under conditions of simple compressive stress. Incompressible deformation is considered with the specimen remaining cylindrical, effects of friction and inertia of the specimen being neglected. A strain rate effect is considered giving the flow stress $\sigma=\sigma_0\log\dot{\epsilon}/\dot{\epsilon}_0$ for $\dot{\epsilon}\geqq e\dot{\epsilon}_0$, $\sigma=\sigma_0$ for $\dot{\epsilon}\leqq e\dot{\epsilon}_0$, where $\dot{\epsilon}$ is the rate of increase of natural strain and σ the true stress. A differential equation is obtained for the variation of the length of the specimen with time after impact. Some simplification is obtained by using nominal strain based on the initial specimen length, and the analysis is developed to include the effects of work hardening. The solution for the final strain is obtained for the case of no work hardening. Numerical results show agreement with experiment, as also does a theory based on a linear, rather than logarithmic, strain rate effect.

E. H. Lee (Providence, R. I.).

Petrašen', G. Lamb's problem for an elastic half-space. Doklady Akad. Nauk SSSR (N.S.) **64**, 649–652 (1949). (Russian)

Un problème simple permet d'exposer une méthode nouvelle d'étude des processus dynamiques dans certains corps élastiques. Cette méthode, contenant comme cas particulier la méthode de solutions complexes de V. Smirnoff et S. Soboleff, permet de résoudre effectivement le problème de processus dynamiques dans un demi-espace (problème de Lamb), dans une couche et dans une sphère élastique sans l'action d'une impulsion concentrée. Les formules obtenues permettent de discerner et d'étudier les ondes de Rayleigh et leur propagation. *V. A. Kostitsin* (Paris).

Šerman, D. I. On the propagation of waves in a fluid layer covering an elastic half-space. Acad. Sci. URSS. Publ. [Trudy] Inst. Séismolog. no. **115**, 43 pp. (1945). (Russian)

Dans ce mémoire l'auteur étudie le problème de détermination des ondes superficielles de Rayleigh sur les limites des milieux considérés. Le problème est effectivement résolu

dans le cas particulier de présence d'une force périodique superficielle appliquée sur la surface de séparation de deux milieux ou sur la limite supérieure de la couche fluide. Les formules obtenues sont discutées, et les oscillations principale et résiduelle, à des distances suffisamment grandes des points d'application de la force périodique superficielle, sont calculées. *V. A. Kostitzin* (Paris).

Koppe, H. Über Rayleigh-Wellen an der Grenzfläche zweier Medien. Z. Angew. Math. Mech. **28**, 355–360 (1948). (German. Russian summary)

The author applies the Rayleigh wave theory to boundary waves between two media. The first case he considers is that of two semi-infinite isotropic elastic solids in welded contact. The irrational equation derived for the velocity of boundary waves between the two solids in terms of the elastic constants and the transverse wave velocities in the two media is obtained on the assumption that a displacement potential exists for motion parallel to the interface and that amplitudes normal to the interface fall off exponentially with distance from the interface in both media. Solving numerically for the case of equality of the two Lamé constants, the author finds that boundary waves exist only if the ratio of the rigidity in the acoustically less dense medium to that in the denser medium lies between 3 and 0, and the boundary wave velocity lies between the velocity of Rayleigh waves and transverse waves in the denser medium. The second case the author considers is that of waves of a fluid in contact with a solid. He concludes that boundary waves can always exist and that they travel more slowly than the corresponding Rayleigh waves. [There are a few minor typographical errors, such as subscript z in place of X in the last term of equation (2) on page 356, and $p=\frac{1}{3}$ instead of $q=\frac{1}{3}$ in line 6 on page 359.]
J. B. Macelwane (St. Louis, Mo.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Herzberger, M. Image error theory for finite aperture and field. I. The image of a point. Geometry of the wave surface. Ottica (N.S.) **1**, 83–87 (1947).

Reprinted from J. Opt. Soc. Amer. **38**, 736–738 (1948); these Rev. **10**, 220.

Di Jorio, Mario. The general theory of isoplanatism for finite aperture and field. J. Opt. Soc. Amer. **39**, 305–319 (1949).

The author gives the differential condition for the fact that an infinitesimal line element is imaged so that to each point there belongs a congruent caustic, in such a way that the images of two neighboring points are obtained by rotation and translation of homologous rays. This condition is more general than the Staeble-Lihotzky condition for isoplanatism and less general than the conditions for homoeoplanatism of Boegehold-Herzberger [see Herzberger, Strahlenoptik, Springer, Berlin, 1931, pp. 169 ff.]. However, the author restricts himself to the consideration of meridional bundles. It is shown that his condition contains many well-known formulae as special cases. [The author errs in assuming that his condition is transitive, i.e., that a system combined of two isoplanatic systems is isoplanatic. This is the case only under very restrictive conditions, as is geometrically evident. Moreover, the general caustic of an off-axis point is more general than the author assumes; it seldom has an axis of symmetry.] *M. Herzberger.*

Rosenthal, G. Die geometrisch-optischen Probleme des Tripelspiegels. Optik **4**, 391–409 (1949).

Wagner, Helmut. Anwendung moderner mathematischer Methoden auf Probleme des optischen Rechnens. Arch. Math. **1**, 303–311 (1949).

The article announces a new method of optical calculation without going into practical detail. The ideas (computation of matrices containing the partial differential quotients of the calculation data with respect to the construction data) fits equally well the methods developed by T. Smith [cf. Trans. Opt. Soc. **25**, 273–282 (1924); Proc. Phys. Soc. **57**, 286–293 (1945)] and the methods published by the reviewer [cf., e.g., Strahlenoptik, Springer, Berlin, 1931, pp. 94 ff.]. The promised detailed publication will tell whether any novelty is involved. *M. Herzberger* (Rochester, N. Y.).

Slevogt, H. Zur Beugungstheorie rotationssymmetrischer Wellen. Optik **4**, 349–355 (1949).

The author discusses a new method for improving an optical system. The diffraction integral, taken over a spherical wave front of given phase and amplitude variations, is developed in a Neumann series of Bessel functions. For practical purposes the reverse procedure is more important. The author suggests a method for calculating phase and amplitude variations of the wave function over the exit aperture, for arbitrarily prescribed intensity distribution in the image plane. *C. J. Bouwkamp* (Eindhoven).

Bremmer, H. Some remarks on the ionospheric double refraction. I. Geometrical optics. Philips Research Rep. **4**, 1–19 (1949).

After recalling the general formulae connecting the refractive index and the polarization of the ordinary and extraordinary waves in the ionosphere with the angle between their normals and the earth's magnetic field, the author investigates in detail the case of short waves whose frequency is much greater than the collision frequency of the electrons. Snell's law is worked out in explicit form, so as to allow the calculation of the elevation angle of the two wave-normals for any given height on the horizon. From the fourth-order differential equation satisfied by each component of the electromagnetic field one can deduce the differential equation of the characteristic surfaces (eiconal equation) and from this latter the equations of the bicharacteristics (rays). In this manner a formula is derived for the direction of the ordinary and extraordinary rays. Both rays are found to be contained in the plane through the wave-normal and the earth's magnetic field. The ray and the normal coincide when they are parallel to the earth's magnetic field as well as when this latter is perpendicular to the plane of incidence on the ionosphere. The same results are proved by means of Fresnel's indicial surface, Fermat's principle and Poynting's vector. Further the author examines the splitting of a given incident ray impinging on the ionosphere into an ordinary and an extraordinary ray and their state of polarization when leaving the

ionosphere after reflection. Finally long waves are dealt with, assuming their reflection to take place suddenly against a homogeneous doubly refracting ionosphere.
G. Toraldo di Francia (Florence).

Sobolev, V. V. On the coefficients of brightness of a plane layer of a scattering medium. Doklady Akad. Nauk SSSR (N.S.) **61**, 803–806 (1948). (Russian)

From the usual equations of radiative transfer and radiative equilibrium the author derives explicit integral equations for the coefficients of brightness in a plane layer of a scattering medium. Numerical solutions of these are given in the case of no absorption and spherically symmetrical scattering, and are compared with solutions obtained by approximate methods. The author also considers the case where the medium is bounded on one side by a reflecting surface of given albedo. *F. Smithies.*

Finzi, Bruno. Formulazione integrale delle leggi elettromagnetiche nello spazio-tempo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 203–211 (1948).

Applying the generalisations of Green's and Stokes' integral theorems in a space of four dimensions to the four-dimensional form of the electromagnetic equations, the author expresses the electromagnetic laws in integral forms.
A. J. McConnell (Dublin).

Kilmister, C. W., and Chirgwin, B. H. A note on minimum integrals in field theory. Philos. Mag. (7) **40**, 226–232 (1949).

It is shown that the minimum energy theorems of electrostatics, current electricity, and incompressible fluids can all be derived from the integral

$$I=\int\int\int(\mathbf{A}\cdot\boldsymbol{\phi}\cdot\mathbf{A}+2\mathbf{C}\cdot\mathbf{A})d\tau$$

where $\boldsymbol{\phi}$ is a prescribed, symmetric dyadic, and $\mathbf{A}$ and $\mathbf{C}$ are certain vectors appropriate for the particular field. The theorem is also applied to magnetostatics and is shown to give results in agreement with those given earlier by G. H. Livens [same Mag. (7) **38**, 453–479 (1947); these Rev. **9**, 258]. *C. Kikuchi* (East Lansing, Mich.).

Durand, Émile. Les développements en séries des grandeurs retardées de l'électromagnétisme classique. J. Phys. Radium (8) **10**, 41–48 (1949).

Calculations of electromagnetic field quantities, based on a method developed by the author [C. R. Acad. Sci. Paris **219**, 510–513 (1944); these Rev. **7**, 401]. *C. Kikuchi.*

Smith-White, W. B. On the mechanical forces in dielectrics. Philos. Mag. (7) **40**, 466–479 (1949).

There exist in the literature two distinct formulas for the mechanical force acting in a dielectric. One is due to Helmholtz [Ann. Physik Chemie (N.S.) **13**, 385–406 (1881)], which is quoted in many texts on electricity, e.g., the standard works of Jeans and Abraham-Becker. Helmholtz's formula for a fluid dielectric of susceptibility k and density τ is

$$\mathbf{F}^{(A)}=-\tfrac{1}{2}\mathbf{E}^2\nabla k+\tfrac{1}{2}\nabla(\mathbf{E}^2\tau\partial k/\partial\tau), \tag{1}$$

where $\mathbf{E}$ is the electric field strength. The other formula is consistent with Kelvin's theory of dielectrics and is general, viz.,

$$\mathbf{F}=(\mathbf{P}\cdot\nabla)\mathbf{E}. \tag{2}$$

In the case of a fluid dielectric the difference between the two expressions is not zero:

$$\mathbf{F}^{(A)}-\mathbf{F}=\tfrac{1}{2}\nabla[\mathbf{E}^2\tau^2\partial/\partial\tau(k/\tau)].$$

The author's theory leads to the conclusion that only (2) is consistent with the principle of energy, while Helmholtz's theory is incompatible with the Poisson-Kelvin views on the cause of the dielectric behavior. Larmor [Philos. Trans. Roy. Soc. London. Ser. A. **190**, 205–300 (1897)] has also criticized expression (1). *C. J. Bouwkamp* (Eindhoven).

Niessen, Karel Frederik. The earth's constants from combined electric and magnetic measurements partly in the vicinity of the emitter. Z. Naturforschung **3a**, 552–558 (1948).

The author discusses a method for determining by field strength measurements the numerical values of the conductance σ and the dielectric constant ϵ of the earth. The method is based on the simple expression of van der Pol and Niessen [Ann. Physik (5) **6**, 273–294 (1930)] for the vector potential of a vertical electric Hertzian dipole located on the surface of the homogeneous plane earth, for points on that surface. The author shows that six relative measurements, relating to the vertical electric field and the horizontal magnetic field, at three different points on the earth's surface, are sufficient to determine σ and ϵ for a fixed wavelength. Two of these points are at distances from the dipole between two and four wave-lengths, the third point is in the far zone. The advantage of the new method over others is discussed. *C. J. Bouwkamp* (Eindhoven).

Hopkins, H. H. The disturbance near the focus of waves of radially non-uniform amplitude. Proc. Phys. Soc. **62**, 22–32 (1949).

Lommel's integral for the intensity near the focus of a spherical wave limited by a circular aperture of radius r is

$$C+iS=2\pi r^2z^{-2}\int_0^r A\exp(-\tfrac{1}{2}iyx^2/z^2)J_0(x)xdx,$$

where A is the wave amplitude and y, z are distance parameters which measure the distance from the focal plane and from the center of the diffraction pattern, respectively. Lommel evaluated the integral in terms of his U and V functions on the assumption of constant amplitude. The present paper generalizes Lommel's results to radially varying amplitudes, assuming $A=Q+P(\rho/r)^2$. It is shown that the integral can still be evaluated in terms of Lommel's functions and certain associated functions. The author includes an extensive series of curves for the intensity in different planes of focus, for various values of Q and P, as well as the corresponding curves for uniform amplitude.
M. C. Gray (Murray Hill, N. J.).

Herpin, André. Réflexion de la lumière sur des milieux isotropes doués de pouvoir rotatoire. Rev. Optique **28**, 65–78 (1949).

The reflection of a plane wave at the surface of a medium possessing natural rotating power may be treated by the classical method of requiring continuity for the tangential components of the field. Let us denote by n_+ and n_- the indices of refraction of the two circularly polarized waves in the second medium (the index in the first medium being 1), with i the angle of incidence and with r_0 the average of the two angles of refraction. If the primary vibration is

perpendicular to the plane of incidence ($E_{\parallel}=0$) one finds for the reflected wave

$$E_{\perp}^{r}=-\frac{\sin(i-r_0)}{\sin(i+r_0)}E_{\perp},$$

$$E_{\parallel}^{r}=-j(n_+-n_-)\frac{\sin^2 i\sin^2 r_0}{\cos r_0(\sin^2 i+\sin^2 r_0)\sin(i+r_0)}E_{\perp}.$$

If the primary vibration is parallel to the plane of incidence ($E_{\perp}=0$), the following formulae hold:

$$E_{\parallel}^{r}=\frac{\sin 2i-\sin 2r_0}{\sin 2i+\sin 2r_0}E_{\parallel},$$

$$E_{\perp}^{r}=-2j\frac{n_+-n_-}{n_0}\frac{\sin i\sin^2 r_0}{\sin(i+r_0)(\sin 2i+\sin^2 r_0)}E_{\parallel}.$$

In both cases the reflected vibration is elliptically polarized. If the second medium is more refractive than the first one, the axes of the ellipse are respectively parallel and perpendicular to the plane of incidence. In the contrary case they are rotated by a certain angle. The author further investigates the incidence near the limiting angle and the propagation of inhomogeneous waves. *G. Toraldo di Francia.*

Godfrey, G. H. Diffraction of light from sources of finite dimensions. Australian J. Sci. Research. Ser. A. **1**, 1–17 (1948).

The distribution of light intensity in the diffraction pattern of an extended light source of uniform brightness is treated analytically, on the assumption that the points comprising the source are non-coherent. After discussing some general results, the author considers the Fraunhofer diffraction at rectangular and circular apertures for the case of a luminous strip. Maxima and minima of light intensity in the image are studied, and the conditions under which they can be observed are indicated. Certain diffraction effects not previously recorded are mentioned, and applications to the optics of the telescope and the spectroscope are discussed. The mathematics used is elementary. Among the tabulated functions the following are noteworthy:

$$10^6[\mathrm{Si}\,(2x)-x^{-1}\sin^2 x]/\pi,$$

$$x=0(0.1)15(0.5)34.5\ [5S],$$

and

$$\int_0^x[H_1(2t)/\pi t^2]dt,\quad x=0(0.1)15\ [4D],$$

where Si and H_1 are the sine integral and Struve's function, respectively. *C. J. Bouwkamp* (Eindhoven).

Magnus, W., und Oberhettinger, F. Über einige Randwertprobleme der Schwingungsgleichung $\Delta u+k^2u=0$ im Falle ebener Begrenzungen. J. Reine Angew. Math. **186**, 184–192 (1945).

If electromagnetic waves are generated by a point source (electric or magnetic dipole) located between two parallel perfectly conducting planes, the total effect can be regarded as due to the original source and its images in the boundary planes. These sources form two linear lattices, and the vector potential can be expressed in terms of sums over the lattices; but the resulting series are not absolutely convergent and only converge slowly. The point of the paper is that the series can be converted into rapidly convergent series by the use of Poisson's summation formula,

$$\sum_{-\infty}^{\infty}F(y+nd)=d^{-1}\sum_{-\infty}^{\infty}e^{-2n\pi iy/d}f(2n\pi/d),$$

where $f(\xi)=\int_{-\infty}^{\infty}e^{i\xi\eta}F(\eta)d\eta$. For example,

$$\sum_{-\infty}^{\infty}\frac{e^{-ik\sqrt{\{x^2+(y-nd)^2\}}}}{\sqrt{\{x^2+(y-nd)^2\}}}=-\pi id^{-1}\sum_{-\infty}^{\infty}H_0^{(2)}(k_n|x|)e^{-2n\pi iy/d},$$

where $k_n=\sqrt{\{k^2-4n^2\pi^2/d^2\}}$. If k is real, only a few of the numbers k_n are real; the rest are complex with negative imaginary part. The method is also applied to a line source between parallel planes and a point source in a rectangular wave guide. *E. T. Copson* (Dundee).

Croze, François, et Darmois, Georges. Réduction à l'unité des expressions du principe de Huyghens pour les ondes électromagnétiques. C. R. Acad. Sci. Paris **228**, 824–826 (1949).

It is pointed out that a formulation of Huyghens' principle for electromagnetic waves is mathematically and physically correct only if it satisfies the following conditions: (i) it must give the correct electromagnetic field of a source Σ at a point P as the resultant of elementary fields generated by secondary sources on a surface S separating P and Σ; (ii) if S is a closed surface, the secondary sources on S must give a null effect at any point Q situated on the same side of S as P; (iii) the secondary waves must be electromagnetic waves. It is stated that many of the proposed formulations of Huyghens' principle do not satisfy these conditions.
E. T. Copson (Dundee).

Franz, Walter. Zur Formulierung des Huygensschen Prinzips. Z. Naturforschung **3a**, 500–506 (1948).

An analytical formulation of Huyghens' principle for electromagnetic waves specified by electric and magnetic vectors $\mathbf{E}e^{i\omega t}$, $\mathbf{H}e^{i\omega t}$ is derived by using a Green's dyadic. This dyadic

$$\Gamma(P,Q)=\mathrm{curl}_P\frac{e^{ikr}}{4\pi r}\mathrm{I}\mu,\qquad r=PQ,$$

where I is the unit dyadic, plays the same part in the theory as does the elementary solution $e^{ikr}/(4\pi r)$ in Helmholtz's theory of the solution of $(\nabla^2+k^2)\mu=0$. The electric force at P due to a magnetic oscillator at Q is $\Gamma(P,Q)\cdot\mathbf{e}$, where $\mathbf{e}$ is a unit vector in the direction of the axis of the oscillator.

The principal result of the paper is (with a slight change of notation) that the expressions

$$\frac{1}{\mu(P)}\iint_S\overline{\Gamma}(J,P)\cdot\mathbf{N}\times\mathbf{H}(J)dS+\frac{1}{i\omega\mu(P)}\,\mathrm{curl}\iint_S\Gamma(P,J)\cdot\frac{1}{\mu(J)}\cdot\mathbf{N}\times\mathbf{E}(J)dS,$$

$$\iint_S\Gamma(P,J)\cdot\frac{1}{\mu(J)}\cdot\mathbf{N}\times\mathbf{E}(J)dS-\frac{1}{i\omega\epsilon(P)}\mathrm{curl}\frac{1}{\mu(P)}\iint_S\overline{\Gamma}(J,P)\cdot\mathbf{N}\times\mathbf{H}(J)ds,$$

where $\mathbf{N}$ is the inward normal unit vector to the closed surface S at the integration point J, represent $\mathbf{H}(P)$ and $\mathbf{E}(P)$, respectively, when P is inside S, but vanish when P is outside S. Thus the field inside P can be regarded as due

to a surface distribution of magnetic dipoles $\mu^{-1}\mathbf{N}\times\mathbf{E}$ and of electric dipoles $\epsilon^{-1}\mathbf{N}\times\mathbf{H}$ on S. These formulae are made the basis of a theory of diffraction of electromagnetic waves analogous to Kirchhoff's theory of diffraction of sound waves. The resulting solutions are interpreted as the solution, not of a boundary value problem, but of a saltus-problem. [Cf. Kottler, Ann. Physik (4) 70, 405–456 (1923); 71, 457–508 (1923).]

It should be pointed out that Croze and Darmois [see the preceding review] claim that the formulae of Franz do not satisfy all the conditions required of a mathematically and physically correct formulation of Huyghens' principle.

E. T. Copson (Dundee).

Keller, Herbert B., and Keller, Joseph B. Reflection and transmission of electromagnetic waves by a spherical shell. J. Appl. Phys. 20, 393–396 (1949).

A dipole is surrounded by a sphere whose electrical characteristics are μ_1, ϵ_1 and σ_1. The dipole is at the center of the sphere. This sphere is in turn surrounded by a second concentric sphere and the space between these two spheres has electrical characteristics μ_2, σ_2 and ϵ_2. The space outside of the second sphere has an ϵ_3, a μ_3 and a σ_3. The authors find the appropriate solution of Maxwell's equations for this configuration. Comparisons with known limiting cases are made.

A. E. Heins (Pittsburgh, Pa.).

Hutter, R. G. E. Electron beam deflection. I. Small-angle deflection theory. J. Appl. Phys. 18, 740–758 (1947).

The author investigates the electron optical properties of electric and magnetic fields which possess planes of symmetry. First he calculates the coefficients of deflection for the third order aberrations for an electrostatic field. He makes use of Hamilton's point characteristic (eiconal method). This problem has been treated in an exhaustive manner by Picht and Himpan [Ann. Physik (5) 39, 409–435, 436–477, 478–501 (1941); these Rev. 4, 289] and also Picht [ibid. 43, 53–72 (1943); these Rev. 5, 163] by the so-called path method (Picard method), and also by Melkich [Akad. Wiss. Wien, S.-B. IIa. 155, 393–438, 439–471 (1947); these Rev. 9, 549]. The coefficients of deflection obtained here agree with the expressions of Picht and Himpan, except for terms which arise from a change of position at the plane P_2 ($z=z_2$), beyond which the field vanishes, this being due to the fact that the eiconal method takes account only of the slope correction at the plane P_2, whereas the path method corrects the first path approximation everywhere in the field region and beyond, up to the image. The coefficients of deflection are expressed as integrals of the field function along the axis of motion. In the next section the deflection due to a magnetic field alone is discussed. The magnetic field considered is one which possesses mirror image properties with respect to the planes (x, z) and (y, z). These pairs of fields, the horizontal deflection field and the vertical deflection field, can be obtained from each other by interchanging the coordinates x and y. By using the path method, the author obtains the deflection coefficients of the two fields acting simultaneously (superimposed fields). The case where either the vertical or the horizontal deflection field acts separately, but at the same time an electric field possessing a plane of symmetry is superimposed, has been treated thoroughly by Wendt [Z. Physik 118, 593–617 (1942); 119, 423–462 (1942)] and briefly by Glaser [ibid. 120, 1–15 (1942); these Rev. 4, 204]. The deflection coefficients are expressed by integral expressions in terms of both fields and their first derivatives along the z-axis. One obtains Wendt's expressions for the coefficients if one neglects either the horizontal or the vertical field provided the terms arising from the electric field are neglected in Wendt's results.

N. Chako (Auburn, Ala.).

Vaĭnshteĭn, L. A. A rigorous solution of the problem of the plane wave-guide with an open end. Izvestiya Akad. Nauk SSSR. Ser. Fiz. 12, 144–165 (1948). (Russian)

The wave guide consists of two perfectly conducting half-planes, extending from $z=0$ to $z=\infty$, their distance apart being d. By taking as unknown the surface density of current j on the two walls and considering the vector potential, one obtains the integral equation

$$\int_0^\infty l(z-\zeta)f(\zeta)d\zeta, \qquad z>0. \tag{1}$$

In the case of the magnetic waves H_{0n}, $f(z)$ represents the value of j_z at the upper wall and the kernel $l(z)$ is given by

$$l(z)=\tfrac{1}{2}\{H_0^{(1)}(k|z|)-(-1)^n H_0^{(1)}(k(d^2+z^2)^{\frac{1}{2}})\}.$$

In the case of the electric waves E_{0n}, $f(z)$ represents the value of j_x at the upper wall (and therefore $f(0)$ must be 0) and the kernel $l(z)$ is given by

$$l(z)=\tfrac{1}{2}(\partial^2/\partial z^2+k^2)\{H_0^{(1)}(k|z|)-(-1)^n H_0^{(1)}(k(d^2+z^2)^{\frac{1}{2}})\}.$$

Let the incident wave, coming from $z=\infty$, have the wave-number $-k$, so that one can write $f(z)=Ae^{-ikz}+\bar{f}(z)$, where $\bar{f}(z)$ is due to the reflected waves. If one assumes for a moment the wave-number k (and consequently those of all the waves inside the guide) to have a vanishingly small imaginary part, one can write the improper Fourier integral:

$$f(z)=\int_C e^{iwz}F(w)dw, \tag{2}$$

the function $F(w)$ having a pole at $w=-k$ and the contour C consisting of the real axis and a loop surrounding $-k$. The integral equation (1) then becomes

$$\int_C e^{iwz}F(w)L(w)dw=0, \qquad z>0,$$

where $L(w)$ is the Fourier transform of $l(z)$. As $f(z)$ must be 0 for $z<0$, Jordan's lemma leads to the requirements that $F(w)$ and $F(w)L(w)$ are holomorphic respectively below and above C and tend to zero for $|w|\to\infty$. The author succeeds in determining as infinite products two functions $L_1(w)$, $L_2(w)$ holomorphic and without zeros respectively above and below C, such that $L(w)=(w^2-k^2)L_1(w)L_2(w)$. The required function $F(w)$ is then evidently

$$F(w)=K/(w^2-k^2)L_2(w)=KL_1(w)/L(w)$$

and by means of (2) one finds $f(z)$, and then the field components. The reflection coefficients and the radiation patterns are calculated in closed form for the incident waves E_{00}, E_{01}, E_{02}, H_{01}, H_{02}. A comparison with Kirchhoff's method shows that this latter gives in general a good approximation, except for the back radiation and the case where the incident wave approaches its critical frequency.

G. Toraldo di Francia (Florence).

Vaĭnšteĭn, L. A. On the theory of diffraction by two parallel half-planes. Izvestiya Akad. Nauk SSSR. Ser. Fiz. **12**, 166–180 (1948). (Russian)

The calculation of the radiation pattern from a plane wave-guide, treated in the preceding paper, becomes very laborious as soon as the width $q=d/\lambda$ of the wave-guide, measured in wave-lengths, exceeds a few units. But one can find asymptotic expressions valid for $q\gg 1$, by applying a method similar to that of steepest descent. For instance, if P is the power carried by the incident wave H_{01}, the power $\Sigma_0(\varphi)d\varphi$ radiated in the direction forming an angle φ with the z-axis is

$$\Sigma_0(\varphi)=P\frac{1}{8\pi^2q^2\gamma}\frac{\sin^2\varphi/2}{\sin^2\varphi_0/2}\left\{\frac{\cos(\pi q\sin\varphi)}{\cos\varphi-\cos\varphi_0}\right\}^2, \qquad \cos\varphi<0,$$

$$\Sigma_0(\varphi)=P\frac{1}{32\pi^2q^2\gamma}\frac{\sin^2\varphi/2}{\sin^2\varphi_0/2}\frac{1}{(\cos\varphi-\cos\varphi_0)^2}, \qquad \cos\varphi>0,$$

where $\gamma=(q^2-\frac{1}{4})^{\frac{1}{2}}$ and $\cos\varphi_0=-h/k$. These limiting formulae are valid everywhere except in a region at right angles with the z-axis and of angular amplitude $d\varphi=(2\pi q)^{-\frac{1}{2}}$. Similar formulae, calculated for the incident waves H_{02}, E_{01}, E_{11}, E_{02}, when compared with the rigorous ones, prove to be valid with a good approximation also for values of q near unity. In every case they lead to a better approximation than those derived by Kirchhoff's method, especially in the back half-space ($\cos\varphi>0$). *G. Toraldo di Francia.*

Rice, S. O. Reflection from corners in rectangular wave guides—conformal transformation. Bell System Tech. J. **28**, 104–135 (1949).

A rectangular wave guide with sharp corners is excited by the dominant mode at infinity. It is required to calculate the reflection coefficient due to this abrupt change in the direction of the guide. The so-called E and H plane polarizations are considered. Following a remark due to Routh [Advanced Rigid Dynamics, 6th ed., pp. 461–467, Macmillan, London, 1930], the author transforms the guide with corners into a strip by conformal transformation. This casts the wave equation from one with constant coefficients to one with variable coefficients. The resulting boundary value problem in transformed coordinates may be expressed as an integral equation and this equation may be solved approximately by successive substitutions. The method may be applied to other types of changes in cross section of rectangular wave guides. The relation to other types of calculations for the above class of problems is discussed.

A. E. Heins (Pittsburgh, Pa.).

Rice, S. O. A set of second-order differential equations associated with reflections in rectangular wave guides—application to guide connected to horn. Bell System Tech. J. **28**, 136–156 (1949).

The author treats the excitation of a rectangular wave guide coupled to a sectoral horn. As in the paper reviewed above, it is desired to find the reflection coefficient from the dominant mode in the guide. The problem is reduced to system of ordinary differential equations. The author extends the W.K.B. method for a single second order equation to a system of equations and approximate expressions for the reflection coefficient are derived. *A. E. Heins.*

Hallén, Erik. Properties of a long antenna. J. Appl. Phys. **19**, 1140–1147 (1948).

The integral equation for a progressive current wave in an infinitely long cylindrical antenna is discussed from the point of view of numerical computations. It is shown that the current amplitude function can be expanded in a series of inverse powers of a new parameter Ω_z, which is a function of the distance along the antenna. The coefficients are expressed in terms of certain integrals associated with the sine and cosine integrals which have been previously tabulated [Trans. Roy. Inst. Tech. Stockholm no. **12** (1947); these Rev. **10**, 151]. The paper also includes a critical comparison with Schelkunoff's solution of the long antenna problem [Proc. I.R.E. **33**, 872–878 (1945); these Rev. **7**, 271]. *M. C. Gray* (Murray Hill, N. J.).

Levin, M. L. The characteristic oscillations of thin metallic and slotted antennas. Doklady Akad. Nauk SSSR (N.S.) **63**, 661–664 (1948). (Russian)

According to the Russian references quoted [Ya. N. Fel'd, Žurnal Techn. Fiz. **17**, 1051–1067 (1947); A. A. Pistolkors, ibid. **14**, 693–702 (1944); M. L. Levin, Izvestiya Akad. Nauk SSSR. Ser. Fiz. **11**, 117–133 (1947)], the general integrodifferential equation for the current distribution function $\psi(s)$ along a linear antenna,

$$\psi''(s)+k^2\psi(s)=ikc\cdot\chi\cdot\{K(s)/I+G(\psi;s;k)\},$$

where $\chi=[2ln(l/a)]^{-1}$ is the Hallén parameter, $K(s)$ the exciting distribution, and G a linear integrodifferential operator, is valid also for slotted aerials provided G is suitably chosen. The conventional perturbation method in terms of the small parameter χ is used for the formal solution of both the forced and eigen-distributions, starting with the sinusoidal distribution $\psi_0(s)$ valid for the ideal wire. First order approximations for the eigen-wavelengths and decay constants of order n are obtained in terms of the corresponding complex characteristic impedances. The problem is generalized to N parallel mutually coupled antennas, leading to corresponding Nth order secular equations.

H. G. Baerwald (Cleveland, Ohio).

Pekeris, C. L. Asymptotic solutions for the normal modes in the theory of microwave propagation. J. Appl. Phys. **17**, 1108–1124 (1946).

The author develops methods for calculating the characteristic modes (values) of electromagnetic waves which are propagated in the atmosphere where the refractive index depends on the height above ground by developing the solutions of the wave equation in asymptotic form. In the first part, asymptotic solutions are obtained for the so-called leaky modes, which are characterized by large imaginary parts ($\beta_m>0$) of the characteristic values $\Lambda_m=\alpha_m+i\beta_m$, comparable in value to the real parts α_m. By employing the method first developed by Langer [Physical Rev. (2) **51**, 669–676 (1937)] in connection with quantum mechanical problems, the asymptotic solutions are expressed in terms of Hankel functions of the second kind of fractional order, whose argument is $u(h)=K\int_{h_1}^{h}[\Lambda_m+y(h)]^{\frac{1}{2}}dh$, where $y(h)$ is the height function of the refractive index and h_1 is a zero of the integrand. The coefficients of the Hankel functions contain inverse powers of the propagation constant K. From the boundary condition, $u(0)$ is expressed as an inverse power series in K. From the leading term of $u(0)$ one could calculate the critical point h_1 and therefore in turn Λ_m. The author has also evaluated the next two terms in the series and hence has obtained a correction term for $\Lambda_m^0=-y(h_1)$. There follows a procedure for evaluating the characteristic values from the phase integral equation, and Λ_m is given explicitly.

In the next section the transitional modes are considered. This case arises when β_m is very small and α_m is approximately equal to $-y(h_0)$, where h_0 is the duct height. The solution is developed around this point instead of h_1, since $h_1 \sim h_0$ for these modes. The results of a lengthy analysis for calculating Λ_m are tabulated and compared with Hartree's exact values obtained from the solution obtained by means of the differential analyzer for the case of the exponential and power models. Good agreement is found. The final section deals with trapped modes, i.e., when β_m is very small and one could set $\Lambda_m = -y(h_1)$. Here Langer's method is applied. The phase integral equation is broken into two integrals, one for the interval $0 < h < h_1$ and the other for the interval $h_1 < h < h_0$, h_1 being complex but lying close to the real axis. The author obtains the connection formula for the two regions. The results of this approximation lead to Furry's phase integral expressions for the highly trapped modes. *N. Chako* (Auburn, Ala.).

Ryazin, P. A. Propagation of radio waves near the earth's surface. Akad. Nauk SSSR. Trudy Fiz. Inst. 3, no. 2, 45–120 (1946). (Russian. English summary)

The central problem is the investigation of the phase structure and velocity of the field of radio waves emitted by a vertical antenna located on the plane surface of the earth. The important point is the knowledge of the vertical component of the electric vector on the surface at distances which are not too close to the dipole. It is sufficient to consider the vertical component of the Hertzian vector Π on the outer side of the surface. The retarded phase of Π and the retarded phase of the vertical component of the electric vector, which is identical with the former phase in the wave zone, may be written as $\omega r/c + \varphi(r/\lambda, 0, 2\sigma/f, \epsilon)$, where c is the phase velocity of propagation in the air. The additional phase which is due to the effect of the earth increases the main linear increment of the retarded phase.

For prescribed properties of the earth the additional phase in the wave zone grows monotonically from a minimal positive value and for $r/\lambda \to \infty$ approaches a constant limit which lies between $\pi/2$ and π. The phase velocity of waves over the surface at sufficient distances from the proximal zone is smaller than the velocity of light in vacuum and approaches the latter asymptotically as the distance from the emitter is increased. At sufficiently large distances from the antenna the phase velocity of the earth wave is constant, equals the velocity in the atmosphere and is thus independent of the properties of the ground. The distribution of the additional phase is investigated in detail; this phase determines the distortion of the equiphase surfaces throughout the complete space over the earth and for various conditions of propagation. At a given distance from the antenna that part of the retarded phase which must be added to the main linear value has a maximal value on the surface of the earth and decreases monotonically as the height of the point of observation is increased, approaching values close to zero.

As the front of the wave moves away from the antenna its lower part, which has a velocity below that of light, lags behind the upper part and thus becomes distorted, losing its ideal spherical nature near the surface. The velocity of the front as a whole asymptotically approaches that of light; as a result beginning at certain distances the wave propagates at high altitudes with its original shape while near the earth it conserves its distorted form.

From the author's summary.

Krasil'nikov, V. A. On the influence of the fluctuations of the index of refraction in the atmosphere on the propagation of ultra-short radio waves. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 13, 33–57 (1949). (Russian)

The fluctuations of temperature in the atmosphere are responsible for fluctuations in the index of refraction and one may investigate their influence on the propagation of ultra-short radio waves. The fluctuations of temperature are assumed to follow the law of A. M. Obuhov [same vol., 58–69 (1949)]: $\overline{(T_1 - T_2)^2} = B^2 \rho_{12}^{\frac{2}{3}}$, ρ_{12} being the distance apart of two points 1, 2 in the field and B a constant depending on turbulence. Only those fluctuations are considered that involve dimensions greater than the wave length and cause a deformation of the wave front, rather than a scattering. Making use of the relation $\epsilon - 1 = Kp/T$ between the dielectric constant, the pressure and the temperature, and starting from the scalar wave equation, the author finds for the fluctuation ψ of the logarithm of the amplitude the formula: $\overline{\psi^2} = 1.3 B \rho K \bar{T}^{-2} L^{\frac{1}{3}} \Lambda^{-\frac{1}{3}}$, where L is the distance apart of the generator and the receiver and Λ a length depending on the correlation existing between the second derivatives of the temperature, calculated at two different points of the field. A similar law is found to hold for the relative fluctuations at two different points at the same distance L from the generator. The fluctuations of the phases are, however, proportional to $L^{\frac{1}{3}}$. *G. Toraldo di Francia* (Florence).

Bouwkamp, C. J. On the mutual inductance of two parallel coaxial circles of circular cross-section. Nederl. Akad. Wetensch., Proc. 51, 1280–1290 = Indagationes Math. 10, 424–434 (1948).

The mutual inductance of two parallel coaxial toroids (radii, R_1, R_2), each of circular cross-section (radii, a_1, a_2) is derived in the form of a doubly infinite series,

$$M = \sum_{n,m=0}^{\infty} M_{n,m} (a_1/R_1)^{2n} (a_2/R_2)^{2m},$$

where the coefficients $M_{n,m}$ are independent of a_1 and a_2. It is well known that for two circular wires of vanishing cross-section the coefficient $M_{0,0}$ can be expressed in terms of complete elliptic integrals. The author obtains an integral formula for the general coefficient $M_{n,m}$, and shows that all the integrals can be evaluated in terms of complete elliptic integrals. Explicit expressions are given for the first few coefficients, $n+m \leqq 2$, and approximate formulas derived when a_1/R_1, a_2/R_2 are small compared with unity.

M. C. Gray (Murray Hill, N. J.).

Guillemin, E. A. Synthesis of RC-networks. J. Math. Physics 28, 22–42 (1949).

The author considers the limitation of linear passive networks containing no inductances. He finds that except for a constant loss factor any transfer characteristic may be approximated in amplitude arbitrarily closely and may be realized physically by an R-C network. The minimum phase characteristic associated with the amplitude characteristic is always obtainable but the limitations on additional phase shift are not given by the present theory. Following the theoretical discussion a number of examples are considered. As a result of his theory the author concludes that in particular when inductive elements are used they need not have low parasitic losses to meet exacting requirements in the design of linear passive networks since indeed, as his theory shows, one can proceed entirely with-

out inductances and compensate for losses by an amplifier with a fixed gain. *N. Levinson* (Cambridge, Mass.).

***Foster, Ronald M. The average impedance of an electrical network.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 333–340. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

The author considers a network made up of branches each of which has a resistance r_i and considers the linear graph of this network. Let S_i be the driving point impedance of each branch. Let E be the number of branches (resistances), V the number of vertices of the linear graph, and P the number of separate nonconnected parts of the graph. Then the author shows, from Kirchhoff's rule, the formula $\sum r_i/S_i = E - V + P$, where i is summed from 1 to E. The right side of the formula is a topological invariant known as the nullity of the graph. *N. Levinson.*

Gavrilov, M. A. The construction of relay-contact schemes with bridge connections. Avtomatika i Telemehanika **9**, 466–479 (1948). (Russian)

Consider a relay network between two points A and B. Let the contacts connected directly to A be $a, b, c, \cdots$ and those connected to B be $u, v, w, \cdots$. Depending on the structure of the intermediate network between these sets of contacts, the entire network frequently contains fewer contacts than its equivalent series-parallel networks. Suppose that it is required to synthesize this network, knowing its structural formula, using as few relay contacts as possible. The first step in the procedure given here is to identify $a, b, c, \cdots$ and $u, v, w, \cdots$. If it can be assumed that the intermediate network contains none of the front or back contacts operated by relays $a, b, c, \cdots$ or $u, v, w, \cdots$, these relays can be found at once by an algorithm based on the observation that every path joining A and B passes through exactly one of $a, b, c, \cdots$ and one of $u, v, w, \cdots$. Examples are given illustrating tricks which may sometimes be used when this assumption does not hold and illustrating special methods of designing the intermediate network.

E. N. Gilbert (Murray Hill, N. J.).

Quantum Mechanics

Viard, Jeannine. Cinématique opératorielle du corps solide rigide. C. R. Acad. Sci. Paris **228**, 746–748 (1949).

Developing further her theory of operational or quantum kinematics, the author investigates the general motion of a rigid body relative to a general or "quantum" frame of reference in this theory. She finds that the relative right or left velocity of any point of such a rigid body is the resultant of a velocity of translation of a point fixed relative to the body, and of a right or left velocity of rotation.

O. Frink (State College, Pa.).

Pryce, M. H. L. The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles. Proc. Roy. Soc. London. Ser. A. **195**, 62–81 (1948).

Difficulties arise in the attempt to define relativistically a centre of mass which will have simple properties with respect to total momentum, Poisson brackets of coordinates, extension to particles interacting through a field, etc. Six possible definitions are discussed. Three of those are considered in detail in flat space-time, viz., (i) q^a (space part $\mathbf{q}$), the mean of coordinates of particles in a particular reference frame weighted with dynamical masses, (ii) X^a, the result of (i) applied in a frame where the total momentum is zero and then Lorentz-transformed to any other frame, (iii) $\bar{q}^a$, the mean of (i), (ii) weighted with total energy and total rest-mass, respectively. The q^a are given in a form suitable for any field system having an energy-momentum tensor and to particles interacting through a field. Poisson-brackets are given in classical and quantal senses involving the total linear momentum $\mathbf{P}$, angular momentum $\mathbf{M}$, $\mathbf{q}$, $\mathbf{X}$, $\bar{\mathbf{q}}$ and the "inner angular momenta" $\mathbf{S} = \mathbf{M} - \mathbf{q} \wedge \mathbf{P}$, etc. The components of $\bar{\mathbf{q}}$ commute. In quantum field-theory difficulties arise in defining position for a single particle by the operators $x\times$, $y\times$, $z\times$. The present scheme, however, can define the mass-centre of a single particle. This is applied to the electron and positron and to particles of spins 0 and $\hbar$ with investigation of the extra demands of a one-particle theory. Special features occur in application to photons on account of the zero rest-mass. Of $\mathbf{q}$, $\mathbf{X}$, $\bar{\mathbf{q}}$ only the relativistically covariant $\mathbf{X}$ is of much interest. The components of $\mathbf{X}$ do not in general commute (they do for particles of zero spin): this leads to an uncertainty of order $\hbar/mc$ in simultaneous measurement. *C. Strachan* (Aberdeen).

Le Couteur, K. J. The indefinite metric in relativistic quantum mechanics. Proc. Roy. Soc. London. Ser. A. **196**, 251–272 (1949).

The author quantizes relativistic wave equations describing particles of arbitrary spin by introducing a state space with an indefinite metric. Transition probabilities are defined as absolute values of a unitary matrix T which is obtained from the solution of the Schrödinger equation and which is interpreted as a "scattering matrix." The variable x^0 which usually is interpreted as the time variable in the wave equation can no longer consistently be interpreted as the observer's time coordinate, but must be regarded as a related mathematical variable because, in formulating the T matrix, the wave functions at $x^0 = 0$ and at $x^0 = t$ have to be mixed up. No explicit discussion of the Lorentz invariance of the proposed theory is given. Therefore, the reviewer is not convinced by the author's statement, "in general, our formalism implies no connexion between spin and statistics." *A. H. Taub* (Urbana, Ill.).

Wessel, Walter. Unendliche Darstellungen der Lorentz-Transformation und das Massenproblem. Z. Naturforschung **3a**, 559–564 (1948).

The author postulates that a particle is described by a momentum four-vector p_μ, a velocity vector u^μ, and an angular momentum tensor $M^{\mu\nu} = -M^{\nu\mu}$. The Hamiltonian is given by solving the equation $u^{*\mu} p_\mu = -m_0 c (I^2 + K^2)^{\frac{1}{2}}$, where $u^{*\mu}$ is a dimensionless vector related to u^μ and I and K are dimensionless invariants associated with $M^{\mu\nu}$. The $M^{\mu\nu}$ and hence I and K are to be determined from the infinite-dimensional representations of the Lorentz group. The author postulates the existence of a certain type of representation of the Lorentz group, and shows by a combination of correspondence principle arguments and approximate solutions of a wave equation derived from the above equation that the energy in the rest system (mass) is quantized. If he fixes two parameters, he obtains the mass varying from 228 m_e to 411 m_e (m_e is the electron mass) in the case of the Einstein-Bose statistics, and from 184 m_e to higher values in the case of a particle of spin $\frac{1}{2}$.

A. H. Taub (Urbana, Ill.).

Bopp, Fritz. Feldmechanische Begründung der Diracschen Wellengleichung. Z. Naturforschung **3a**, 564–573 (1948).

The author assumes that the equations of motion, including retardation effects, for a charged particle in an external field are derivable from the variational principle $\delta \int L_0 ds = 0$, where s is the proper time,

$$L_0 = (e^2/c^2) \int u_\alpha(s) u^\alpha(s-\tau) f(\sigma) d\tau - (e/c) u_\alpha(s) \varphi^\alpha(z),$$

the particle path in space-time is given by $z^\alpha = z^\alpha(s)$, $\alpha = 1, 2, 3, 4$,

$$u^\alpha = \dot{z}^\alpha(s), \quad \sigma = c^{-2}[z_\alpha(s) - z_\alpha(s-\tau)][z^\alpha(s) - z^\alpha(s-\tau)],$$

and $f(\sigma)$ is a function which is to be determined. It is assumed that the motion is such that L_0 may be approximated by $L_0 = -m_0 c^2 F(Q) - (e/c) u^\alpha \varphi^\alpha$, where $F(Q) = Q^{\frac{1}{2}}$, $Q = (e^2/c^4) \dot{u}_\alpha \dot{u}^\alpha$, as the velocity of the particle approaches the velocity of light. The remainder of the paper assumes that the velocity is near this value. The equations of motion of the particle are then derived and shown to depend on the velocity and the acceleration. These are written in canonical form by introducing the momentum p as the variable conjugate to position and a variable s as the conjugate to velocity v.

The transition to quantum mechanics is made by treating the momentum and position variables as usual. The commutation relations between v and s are obtained in the usual manner from the Poisson bracket relation satisfied by these variables, where this bracket is defined to be

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial s^i} \frac{\partial B}{\partial v^i} - \frac{\partial A}{\partial v^i} \frac{\partial B}{\partial s^i} \right)$$

for any functions A and B of the variables v^i and s^i. Thus the variables x, p and v, s are treated independently. It is then shown that the Hamiltonian leads to the Dirac equation for an electron if the wave function is assumed to be a linear function of some "spintype variables." It is stated that if the wave function is homogeneous of higher degree in these variables, then the equation described is that representing a particle of higher spin. It is stated that if the velocity v is not near the velocity of light, then the resulting equations describe heavier particles. *A. H. Taub.*

Schwinger, Julian. Quantum electrodynamics. II. Vacuum polarization and self-energy. Physical Rev. (2) **75**, 651–679 (1949).

"The covariant formulation of quantum electrodynamics, developed in a previous paper [Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] is here applied to two elementary problems, the polarization of the vacuum and the self-energies of the electron and photon. In the first section the vacuum of the non-interacting electromagnetic and matter fields is covariantly defined as that state for which the eigenvalue of an arbitrary time-like component of the energy-momentum four-vector is an absolute minimum. The second section treats the induction of a current in the vacuum by an external electromagnetic field. The modification of the matter field properties arising from interaction with the vacuum fluctuations of the electromagnetic field is considered in the third section. No real processes are produced by the first order coupling between the fields. Accordingly, alternative equations of motion for the state vector are constructed, from which the first order interaction term has been eliminated and replaced by the second order coupling which it generates. The latter includes the self action of individual particles and light quanta, the interaction of different particles, and a coupling between particles and light quanta. The light quantum self-energy term is shown to vanish, while that for a particle has the anticipated form for a change in proper mass. It is verified that the energy and momentum modifications produced by self-interaction effects are entirely accounted for by the addition of the electromagnetic proper mass to the mechanical proper mass, an unobservable mass renormalization." From the author's summary.

In this paper the basic mathematical techniques which the author has worked out for handling practical problems in covariant electrodynamics are explained with great thoroughness and attention to detail. A refined technique is required, because the theory necessitates the evaluation of 4-dimensional integrals with integrands which become infinite in a highly singular way at the origin. In dealing with these integrals, the author resolves mathematical ambiguities by making use of the inherent formal properties, such as Lorentz covariance, which the theory possesses.

F. J. Dyson (Birmingham).

***de Broglie, Louis. Mécanique Ondulatoire du Photon et Théorie Quantique des Champs.** Gauthier-Villars, Paris, 1949. vi+208 pp.

From the preface: La théorie quantique des champs électromagnétiques a dû son origine aux travaux de Jordan et de Pauli et Heisenberg [Z. Physik **56**, 1–61 (1929)]; on en trouvera d'excellents exposés dans les livres de Heitler [The Quantum Theory of Radiation, Oxford, 1936] et de Wentzel [Einführung in die Quantentheorie der Wellenfelder, Deuticke, Wien, 1943; these Rev. **9**, 556]. Nous plaçant à un point de vue assez différent, nous avons développé depuis 1934 une mécanique ondulatoire du photon qui a l'avantage de bien montrer comment la théorie de la lumière vient trouver sa place dans le cadre général de la mécanique ondulatoire, tout en permettant de retrouver par l'application directe de la seconde quantification la plupart des résultats essentiels de la théorie quantique des champs. Après avoir ébauché cette théorie, nous en avons fait un exposé d'ensemble dans un ouvrage en deux volumes intitulé "Une nouvelle théorie de la lumière: la mécanique ondulatoire du photon" [v. 1, 2, Hermann, Paris, 1940, 1942], puis dans un autre ouvrage intitulé "Théorie générale des particules à spin" [Gauthier-Villars, Paris, 1943], où nous avons rattaché la mécanique ondulatoire du photon à la théorie générale des particules douées de spin. Aujourd'hui, il nous paraît intéressant de reprendre, pour les approfondir, les compléter et parfois les rectifier, les résultats exposés dans le premier des deux ouvrages que nous venons de citer, de façon notamment à bien mettre en évidence ce qui distingue notre mécanique ondulatoire du photon, de la théorie quantique des champs électromagnétiques telle qu'elle est usuellement exposée.

The book is in three sections. The first applies the general principles of wave-mechanics to a single particle, and in particular to a photon; the second extends the theory to many-particle systems, using with minor variations the usual methods of second quantization; the third deals with photons in interaction with matter. The main difference of the author's method from that of quantum field-theory is the following. Instead of beginning the theory with electromagnetic field-strengths as dynamical variables represented by operators, he uses as primitive idea the wave-function

of a photon, which is a 16-component quantity transforming under Lorentz transformations like the product of two Dirac spinors. Field-strengths are introduced later, as matrix-element-densities which when integrated over space give matrix elements for the annihilation of photons in various states. Field-strengths are not Hermitian operators, the Hermitian field-strength of orthodox theory being essentially the sum of the author's field-strength and an adjoint operator which creates photons instead of annihilating them. In general the author's theory is only formally different from quantum field-theory, and leads to the same results in all specific applications. In particular, the divergence difficulties of electrodynamics are not alleviated.

F. J. Dyson (Birmingham).

Finkelstein, R. J. On the quantization of a unitary field theory. Physical Rev. (2) **75**, 1079–1087 (1949).

For this strictly unitary theory the field quantities are everywhere finite, single-valued and continuous. Particles are small regions where the energy-momentum and charge-current of the field are concentrated: their equations of motion follow from the field equations. In one application of the classical form of the theory the Lagrangian is that of a scalar meson field coupled to the electromagnetic field. The field equations are of course nonlinear. In the quantized theory commutation relations for the field components together with definitions of energy momentum, etc., produce commutation relations for quantities associated with the particles. The latter relations have the usual forms for momentum, angular momentum and (nonrelativistic) position. A relativistic definition of position leads to noncommuting positional coordinates with a corresponding limit of observation of position given by the Compton wavelength of the particle. Equations of motion, and rest-mass and electric charge which are constants of the motion, are derived.

C. Strachan (Aberdeen).

Bureau, Florent. Sur la mécanique ondulatoire de l'électron. C. R. Acad. Sci. Paris **228**, 465–467 (1949).

The author makes the statement that the Dirac equations and those of Maxwell are fundamentally different in that Huyghens' principle holds for the latter and not for the former, but does not justify this statement.

A. H. Taub (Urbana, Ill.).

Utiyama, Ryoyu. On the classical theory of the electron. I. Progress Theoret. Physics **3**, 114–123 (1948).
Utiyama, Ryōyū. On the classical theory of the electron. II. Progress Theoret. Physics **3**, 219–228 (1948).

The main part of these two papers is a treatment of classical theory of the interaction of the electromagnetic field with the electron field; the treatment is relativistic and the part of the Lagrangian referring to the electromagnetic field is somewhat modified. The last section deals with point electrons and gives results already obtained by Dirac [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)] and others.

C. Strachan (Aberdeen).

McManus, H. Classical electrodynamics without singularities. Proc. Roy. Soc. London. Ser. A. **195**, 323–336 (1948).

A Lorentz invariant theory of a particle with an extended charge presents some conceptual difficulties, since the charge must be extended in time as well as in space. In this paper such a theory is constructed, describing a classical extended electron in interaction with the electromagnetic field; various consequences are investigated and found to be physically reasonable, and the electron self-energy and self-force are well-defined and finite. The same theory has been considered independently by Blohinčev [Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fyz. **18**, 566–574 (1948); these Rev. **10**, 345]. Two facts are noted concerning the non-physical "run-away" solutions of the equations of motion. (i) It is not known whether for any choice of the shape of the charge-distribution such solutions can be entirely avoided. (ii) If the mechanical mass of the electron is negative (self-energy greater than observed mass) then such solutions always occur.

F. J. Dyson (Birmingham).

Schönberg, Mario. The Hamiltonian formalism of relativistic dynamics. Summa Brasil. Phys. **1**, 51–90 (1947). (English. Portuguese summary)

Lopes and Schönberg [Physical Rev. (2) **67**, 122–123 (1945)] and Schönberg [ibid., 122 (1945)] have shown that the Lorentz-Dirac equations of motion of a point electron may be derived by considering the retarded electromagnetic field due to a particle to be the sum of the attached field $\frac{1}{2}(F^{\mu\nu}_{\text{ret}}+F^{\mu\nu}_{\text{adv}})$ and the radiated field $\frac{1}{2}(F^{\mu\nu}_{\text{ret}}-F^{\mu\nu}_{\text{adv}})$ where the suffixes refer to the retarded and advanced fields, and it is supposed that the radiated field alone reacts on the generating particle although both fields act on other charged particles. Schönberg [Summa Brasil. Math. **1**, no. 5, 41–75 (1946); no. 6, 77–114 (1946); these Rev. **8**, 427] has generalised this by using $+1$, -1, 0 times the above radiated field to correspond to nonstationary motions of positive, negative energy and to stationary motions respectively. In the present paper he now develops a classical relativistic theory of this type. No applications are given and it is suggested that the theory may be too complicated for the solution of particular problems. It uses neither the λ-limiting process nor the Wentzel field. The Hamiltonian formulation, canonical transformations, various generalisations of the Hamilton-Jacobi equation, and integral invariants are discussed; "average" conservation laws are shown to exist for quantities such as momentum and angular momentum in the stationary motions of isolated systems of particles.

C. Strachan (Aberdeen).

Schönberg, Mario. Quantum theory of the point electron. II. Anais Acad. Brasil. Ci. **20**, 223–245, 309–331 (1948).

Difficulties encountered in previous work [Schönberg, same Anais **18**, 297–339 (1946); these Rev. **8**, 615] arising from several time-variables for point electrons are avoided here by using a relativistic quantised electron wave-field. Relativistic actions at a distance are used and no specific variables are used to describe the electromagnetic field. A c-number theory gives conservation of charge but, even without electromagnetic radiation and external forces, gives only conservation "on the average" of energy and momentum. In the q-number formulation the usual commutation rules and the field equations are not necessarily consistent. Even average conservation of energy in the above circumstances demands extra conditions not obviously satisfied. Stationary states cannot be introduced directly. The theory is modified to overcome the last difficulty and methods of approximate solution are discussed. Difficulties arise, e.g., in radiation reaction, which cannot be met without giving to the radiation field its own degrees of freedom.

C. Strachan (Aberdeen).

Green, H. S. The relativistic quantum mechanics of the elementary particles. Proc. Cambridge Philos. Soc. **45**, 263–274 (1949).

The author defines a relativistic "density matrix" as an operator $\rho(x, t; x', t')$, a function of two complete sets of coordinate variables satisfying an equation of the type (1) given below and an unspecified "boundary condition." Equations of conservation of energy and momentum and the equation of continuity are then derived in terms of this density matrix. The equation satisfied by the density matrix is postulated to be

$$(1) \qquad (\mathbf{p}\cdot\mathbf{A}+(iE/c)A_4+m_1cA_5-im_2c)\rho=0,$$

where A_α ($\alpha=1, \cdots, 5$) are given by $A_\alpha=\sum_{i=1}^{n}\alpha_\alpha^{(i)}$ and the $\alpha_\alpha^{(i)}$ are mutually commuting sets of Dirac matrices, m_1 and m_2 are constants related to the mass of the particles described by the density matrix, and

$$p_i=-i\hbar\frac{\partial}{\partial x^i}\delta(\mathbf{x}-\mathbf{x}'), \quad E=i\hbar\frac{\partial}{\partial t}\delta(t-t').$$

The major part of the paper is concerned with discussing representations of the ring generated by the A_α. The author uses the well-known method of determining a complete set of orthogonal idempotents. *A. H. Taub* (Urbana, Ill.).

Tamm, I. E. On certain mathematical methods of the theory of the scattering of particles. II. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 74–77 (1949). (Russian)

It is observed that the variation method presented in the author's previous paper with the same title [same Zhurnal **18**, 337–345 (1948); these Rev. **9**, 558] is actually valid only in a quite restricted class of cases, because the function $u_l(r)$ which is used has singularities that may make the integral divergent. A more generally valid method can be obtained by using a function $\chi_l(r)$ that is finite for all real positive r. This is illustrated in detail only for the case $l=0$, with $\chi_0(r)$ defined by $\varphi(r)=c\{e^{ikr}-\chi_0(r)e^{-ikr}\}$, where c is a constant. *W. H. Furry* (Cambridge, Mass.).

Molière, Gert. Zur Theorie der Luftschauer. Die mittleren Quadrate der räumlichen und Winkelablenkung. Z. Physik **125**, 250–268 (1949).

Angular and lateral spread of both particles and photons in an air shower are investigated. The spread thus obtained is of the same order as found by Bethe and also by Nordheim, but it is much larger than was calculated by Euler and Wergeland. The discrepancy is shown to arise largely from the fact that the earlier generations of a shower particle contribute considerably to its spread while Euler and Wergeland considered the last generation only. Most of the calculation is concerned with the root mean square value of the spread. The higher moments of the distribution are considered briefly in the last section. All the calculations are based on the approximate cross-sections for collision radiation and pair production introduced originally by Bhabha and Heitler. *L. Jánossy* (Dublin).

Faure, Robert. Correspondance mécanique classique-mécanique ondulatoire. Intégrale du deuxième ordre indépendante du temps. Principe d'extrémum. Formation des opérateurs du deuxième ordre. C. R. Acad. Sci. Paris **227**, 261–263 (1948).

Continuation of a previous note [same C. R. **224**, 797–798 (1947); these Rev. **9**, 8].

Costa de Beauregard, Olivier. Covariance relativiste en théorie du photon superquantifié. C. R. Acad. Sci. Paris **228**, 296–298 (1949).

Kwal, Bernard. Équation d'onde d'un corpuscule de spin 1/2 dans un champ de corpuscule de spin 1 et une généralisation de la transformation de jauge de première espèce. C. R. Acad. Sci. Paris **228**, 980–981 (1949).

Kwal, Bernard. Les champs potentiels et la transformation de jauge généralisée en théorie des corpuscules à spin. J. Phys. Radium (8) **10**, 189–194 (1949).

Petiau, Gérard. Sur l'équation d'ondes du corpuscule de spin total maximum $\frac{h}{2\pi}$ possédant plusieurs états de masse. J. Phys. Radium (8) **9**, 245–248 (1948).

Petiau, Gérard. Sur une extension de la théorie du corpuscule de spin $h/2\pi$ permettant de représenter un méson possédant plusieurs états de masse. C. R. Acad. Sci. Paris **227**, 825–827 (1948).

Petiau, Gérard. L'introduction de coordonnées supplémentaires et la réduction du nombre des constantes d'interactions dans la théorie de la particule de spin $\frac{h}{2\pi}$ et $2\frac{h}{2\pi}$ (gravitation). Disquisit. Math. Phys. **6**, 235–241 (1948).

Petiau, Gérard. Sur l'approximation de l'optique géométrique dans la mécanique ondulatoire du corpuscule de spin $\frac{1}{2}\cdot\frac{h}{2\pi}$ représenté par l'équation d'ondes de Dirac. Ann. Physique (12) **4**, 218–246 (1949).

Petiau, Gérard. Sur les équations d'ondes de la théorie du corpuscule de spin $\frac{h}{2\pi}$ et leurs généralisations. J. Phys. Radium (8) **10**, 215–224 (1949).

Potier, Robert. Sur les équations d'onde du corpuscule de spin 1/2 à masses multiples. C. R. Acad. Sci. Paris **227**, 1146–1147 (1948).

Potier, Robert. Sur la théorie générale des corpuscules de spin total donné. C. R. Acad. Sci. Paris **228**, 656–658 (1949).

Valatin, Jean G. Sur le formalisme de la théorie de l'électron négatif et positif. C. R. Acad. Sci. Paris **228**, 1205–1206 (1949).

Valatin, Jean G. L'opérateur de l'énergie d'interaction électromagnétique de particules chargées. C. R. Acad. Sci. Paris **228**, 1283–1284 (1949).

Valatin, Jean G. Sur l'interaction relativiste des particules et la théorie quantique des champs. C. R. Acad. Sci. Paris **228**, 1329–1331 (1949).

Thermodynamics, Statistical Mechanics

Buchdahl, H. A. On the unrestricted theorem of Carathéodory and its application in the treatment of the second law of thermodynamics. Amer. J. Phys. 17, 212–218 (1949).

The author gives (1) a heuristically simple proof of Carathéodory's theorem concerning the integrability of a total differential equation, and (2) a brief but porous development of the concepts of absolute temperature and entropy. *C. C. Torrance* (Annapolis, Md.).

Tatevskiĭ, V. M., and Frost, A. V. On some transformations of statistical expressions for thermodynamical functions and the constant of equilibrium of reactions. Vestnik Moskov. Univ. 1947, no. 12, 113–119 (1947). (Russian)

The standard expression for the partition sum of ideal polyatomic gases is brought into a different form convenient for some purposes. *L. Tisza* (Cambridge, Mass.).

***Khinchin, A. I. Mathematical Foundations of Statistical Mechanics.** Translated by G. Gamow. Dover Publications, Inc., New York, N. Y., 1949. viii+179 pp. $2.95.

The Russian edition [1943] has already been reviewed [these Rev. 8, 187]. A remark by the author on the relation of statistical mechanics to dialectical materialism has been omitted from the translation. *J. L. Doob* (Ithaca, N. Y.).

Schlüter, Arnulf. Zur Statistik klassischer Gesamtheiten. Z. Naturforschung 3a, 350–360 (1948).

Ergodic systems in which the energy is separable are discussed. Instead of restricting the problem to the case of very many systems, emphasis is placed on deriving exact results for an arbitrary number of particles present. It is shown that the usual thermodynamical quantities (temperature, entropy) may be formally introduced even in the case of a small number of systems, the introduction, however, being dependent on the fact that the system is rigorously ergodic. These thermodynamical quantities are defined in terms of the canonically invariant phase volume of the system, the definitions being very analogous to the usual ones of ordinary statistical mechanics. The formulas obtained for the various quantities under consideration are written elegantly in terms of a "Faltung" product.

J. M. Luttinger (Zurich).

Vlasov, A. A. A new formulation of the many particle problem. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 840–856 (1948). (Russian)

The behavior of N particles is described with the aid of distribution functions $f(\mathbf{r}, \mathbf{V}, t)$ which depend on the coordinates, velocities and time, and satisfy generalized equations of continuity previously proposed by the author [Acad. Sci. USSR. J. Phys. 9, 130–138 (1945); these Rev. 7, 183] as a generalization of the classical N-body problem. In this theory spatial localization of particles in the classical sense has been given up, so that the theory is claimed to occupy an intermediate position between the classical and the quantum formulations of the problem. Mathematically the stationary state problem reduces to the eigenvalue problem of certain nonlinear integral equations whose solutions are discussed in some detail. *G. M. Volkoff* (Vancouver, B. C.).

Domb, C. Order-disorder statistics. I. Proc. Roy. Soc. London. Ser. A. 196, 36–50 (1949).

An alternate form is derived for the well-known matrix method of Kramers and Wannier [Physical Rev. (2) 60, 252–262, 263–276 (1941); these Rev. 3, 63, 64].

L. Tisza (Cambridge, Mass.).

Hartmann, Hermann. Zur statistischen Mechanik der Ordnungs-Unordnungs-Umwandlungen. Z. Naturforschung 3a, 617–619 (1948).

Computation of the partition sum of a one-dimensional nearest neighbor system [first computed by Ising, Z. Physik 31, 253–258 (1925); then (more elegantly than in the present paper) by Kramers and Wannier, Physical Rev. (2) 60, 252–262, 263–276 (1941); these Rev. 3, 63, 64].

L. Tisza (Cambridge, Mass.).

Gurov, K. P. The kinetical equation in quantum mechanics. Vestnik Moskov. Univ. 1947, no. 1, 135–140 (1947). (Russian. English summary)

Bogolyubov's method of deriving the classical kinetic equation [N. N. Bogolyubov, Dynamical Theories in Statistical Physics, G.T.T.I., 1946] has been transcribed into a quantum mechanical form. *L. Tisza.*

Münster, Arnold. Über die statistische Mechanik regulärer und irregulärer Lösungen. Z. Naturforschung 3a, 158–172 (1948).

Approximate calculation of the partition sum of solutions, applied, in particular, to solutions of high polymers.

L. Tisza (Cambridge, Mass.).

Ugarov, V. A. "Theory of strata." Vestnik Moskov. Univ. 1947, no. 7, 159–165 (1947). (Russian)

Cf. Akad. Nauk. SSSR. Zhurnal Eksper. Teoret. Fiz. 18, 457–461 (1948); these Rev. 10, 92.

Jaffé, George. A statistical theory of liquids. III. Physical Rev. (2) 75, 184–196 (1949).

The general kinetic equation is solved by using a series of simplifying assumptions. Numerical values are obtained for the heat conductivity and viscosity of ten liquids with the help of the intermolecular potentials previously used by the author [same Rev. (2) 62, 463–476 (1942); 63, 313–321 (1943); these Rev. 4, 208; 5, 56] for obtaining the equilibrium constants of these liquids. *L. Tisza.*

Kaempffer, Friedrich. Zur Theorie des idealen Bose-Einstein-Gases. Z. Physik 125, 359–369 (1949).

The well-known condensation phenomenon of an ideal Bose-Einstein gas is investigated assuming the relativistic relation between energy and momentum. The theory contains the parameter $\alpha = mc^2/KT$, where mc^2 is the rest energy of a molecule. Explicit expressions are obtained for $\alpha \gg 1$ and $\alpha \ll 1$. *L. Tisza* (Cambridge, Mass.).

Bogolyubov, N. N. Energy levels of a non-ideal Bose-Einstein gas. Vestnik Moskov. Univ. 1947, no. 7, 43–56 (1947). (Russian)

Simplified and more complete derivation of the results of an earlier paper of the author [Acad. Sci. USSR. J. Phys. 11, 23–32 (1947); these Rev. 9, 168]. *L. Tisza.*